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ELECTROMAGNETIC FIELDS OF A UNIFORM SPHERE IN A UNIFORM CONDUCTING MEDIUM WITH APPLICATION TO DIPOLE SOURCES

W. M. WYNN

NCSC NAVAL COASTAL SYSTEMS CENTER
PANAMA CITY, FLORIDA 32407-5000

CAPT D. P. FITCH, USN
Commanding Officer

MR. TED C. BUCKLEY
Technical Director

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Vector spherical harmonic expansions are developed for the electromagnetic field of a uniform sphere in a uniform conducting medium in the presence of an arbitrary localized distribution of current. The source is either interior to or exterior to the sphere. Explicit expressions then are developed for all cases for magnetic and current dipole sources, and a complete numerical treatment is given for the case of exterior dipole and field point.

The general treatment includes the transverse electric-transverse magnetic representation of the electromagnetic field in a source-free region, and demonstration of the formation of the electric and magnetic field vectors from $\mathbf{E} \cdot \mathbf{r}$ and $\mathbf{B} \cdot \mathbf{r}$ alone. General expressions are given relating the scattered field expansion coefficients to the source expansion coefficients, including the special case of a perfectly conducting sphere.

The specialized treatment of dipole sources includes explicit expressions for the electric and magnetic field components both interior to and exterior to the sphere, for either internal or external dipoles, for the general ac case and for the dc limit.

The detailed treatment of the dipole sources includes various series and asymptotic representations of the Bessel functions J_ν and $H_\nu^{(1)}$, which are incorporated into Hewlett-Packard Basic 3.0 codes for the calculation of the dipole field components for the ac and dc cases for exterior dipole and field point.

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INTRODUCTION

The analysis of electromagnetic fields in uniform media reduces to the solution of the vector Helmholtz equation. This equation is a vector partial differential equation of the form $\nabla \times \nabla \times \mathbf{A} - k^2\mathbf{A} = 0$, and generally leads to coupled equations for the field components, but it can be separated in the spherical and cylindrical coordinate systems.[1] For cylindrical coordinates, detailed treatments can be found for localized sources in the presence of a layered cylinder,[2] and for localized sources outside a perfectly conducting wedge,[3] as well as less detailed treatments of the conducting wedge.[4 5] In the present work, a detailed analysis is given for the electromagnetic fields inside and outside a uniform sphere immersed in a uniform medium, in the presence of a localized source either interior or exterior to the sphere.

The solution is given in terms of a vector spherical harmonic expansion, with expansion coefficients for the scattered fields expressed in terms of the coefficients appropriate to the source distribution. The source is treated using a clever technique due to Jackson.[6] A separate treatment of the case of a perfectly conducting sphere also is included.

For explicit results, the source is taken to be a magnetic or current dipole. Without loss of generality, the dipoles are placed on the z-axis, and oriented radially or tangentially to the sphere, which is centered on the origin. Detailed expressions are given for the electric and magnetic field components, both interior and exterior to the sphere, for both interior and exterior dipoles. These expressions are relatively simple, and do not involve any recursion formulas for the coefficients, contrary to what has been reported elsewhere.[7] For completeness, the dc limits of the expressions also are given. This is useful for direct applications, to provide more transparent forms to check for physical consistency of the field expressions, and to provide a numerical check against the codes for the fields in the low-frequency limit.

Numerical codes written in Hewlett-Packard BASIC, Version 3.0, are given for both the ac and dc field expressions for the single case of exterior dipole and field point. The bulk of the code given, in particular the geometrical transformations and the Bessel function and Legendre polynomial routines, is applicable to the other three cases. The numerical treatment of the spherical Bessel functions incorporates several representations found in Abramowitz and Stegun.[8]

BACKGROUND

For a time-harmonic source $e^{-i\omega t}$, Maxwell's equations for a uniform medium take the form

$$\nabla \times \mathbf{E} = i\omega\mathbf{B} \tag{1}$$

$$\nabla \times \mathbf{B} = \mu((\sigma - i\omega\varepsilon)\mathbf{E} + \nabla \times \mathbf{M}_S + \mathbf{J}_S) = \mu\mathbf{J} \tag{2}$$

$$\nabla \cdot (\varepsilon\mathbf{E}) = \rho \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4}$$

where σ, ε, and μ are the medium conductivity, permittivity, and permeability, respectively, and $\mathbf{M}_S$ and $\mathbf{J}_S$ are source magnetization and current density. The sources are included in a general way, but will be specialized to point magnetic and current dipoles for explicit calculations.

The equation of continuity is

$$\nabla \cdot \mathbf{J} - i\omega\rho = 0 \tag{5}$$

that, upon substitution of Equations (2) and (3), can be written

$$\nabla \cdot \left(\mathbf{E} + \frac{\mathbf{J}_S}{\sigma - i\omega\varepsilon}\right) = 0. \tag{6}$$

With the introduction of the divergenceless vector

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{J}_S}{\sigma - i\omega\varepsilon} \tag{7}$$

that is identical to $\mathbf{E}$ in a source-free region, Equations (1) and (2) can be written

$$\nabla \times \mathbf{E}' = i\omega\mathbf{B} + \frac{\nabla \times \mathbf{J}_S}{\sigma - i\omega\varepsilon} \tag{8}$$

and

$$\nabla \times \mathbf{B} = \mu((\sigma - i\omega\varepsilon)\mathbf{E}' + \nabla \times \mathbf{M}_S). \tag{9}$$

With the introduction of the complex wave number

$$k^2 = \mu(i\omega\sigma - \omega^2\varepsilon) \tag{10}$$

Equations (8) and (9) can be written

$$\nabla \times \mathbf{E}' = i\omega\left(\mathbf{B} + \frac{\nabla \times \mathbf{J}_S}{k^2}\right) \tag{11}$$

and

$$\nabla \times \mathbf{B} = \frac{k^2}{i\omega}\mathbf{E}' + \mu\nabla \times \mathbf{M}_S. \tag{12}$$

Applying the vector identity $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$ to Equations (11) and (12) then results in the vector equations with sources given by

$$(\nabla^2 + k^2)\mathbf{E}' = -i\omega\mu\left(\frac{\nabla \times \nabla \times \mathbf{J}_S}{k^2} + \nabla \times \mathbf{M}_S\right) \tag{13}$$

and

$$(\nabla^2 + k^2)\mathbf{B} = -\mu(\nabla \times \nabla \times \mathbf{M}_S + \nabla \times \mathbf{J}_S). \tag{14}$$

In a source free region, the fields $\mathbf{E}'$ and $\mathbf{B}$, satisfying the associated homogeneous forms of Equations (13) and (14), can be expanded in terms of transverse electric and transverse magnetic parts in the form

$$\mathbf{E}' = \nabla \times (\mathbf{r}\Psi_E) + \alpha\nabla \times \nabla \times (\mathbf{r}\Psi_M) \tag{15}$$

and

$$\mathbf{B} = \nabla \times (\mathbf{r}\Psi_M) + \beta\nabla \times \nabla \times (\mathbf{r}\Psi_E) \tag{16}$$

where $\mathbf{r}$ is the radius vector, provided that Ψ_E and Ψ_M satisfy the scalar Helmholtz equation

$$\nabla^2\Psi_E + k^2\Psi_E = 0 \tag{17}$$

and

$$\nabla^2\Psi_M + k^2\Psi_M = 0. \tag{18}$$

This may be verified by applying the operator $\nabla \times \nabla \times (\cdots)$ to Equations (15) and (16) in source free regions, and applying the appropriate vector identities.

The coefficients α and β can be determined by inserting Equations (15) and (16) in Equation (11) in a source free region and using the identity

$$\nabla \times \nabla \times (\mathbf{r}\Psi) = k^2\mathbf{r}\Psi + \nabla\left(\Psi + r\frac{\partial\Psi}{\partial r}\right) \tag{19}$$

to give the result

$$\alpha = \frac{i\omega}{k^2} \quad \text{and} \quad \beta = -\frac{i}{\omega}. \tag{20}$$

Equation (19) is a nontrivial identity, and is developed in detail below Equation (69) and preceding discussion).

The vectors $\mathbf{E}'$ and $\mathbf{B}$ can be determined from the projections $\mathbf{r} \cdot \mathbf{E}'$ and $\mathbf{r} \cdot \mathbf{B}$, as can be seen by constructing the projections using Equations (15) and (16), applying Equation (19) and the identity $\mathbf{r} \cdot (\nabla\Psi \times \mathbf{r}) = 0$. This results in

$$\mathbf{r} \cdot \mathbf{E}' = \frac{i\omega}{k^2}\left[k^2r^2\Psi_M + r\frac{\partial}{\partial r}\left(\Psi_M + r\frac{\partial\Psi_M}{\partial r}\right)\right] \tag{21}$$

and

$$\mathbf{r}\cdot\mathbf{B} = -i\omega\left[k^2r^2\Psi_E + r\frac{\partial}{\partial r}\left(\Psi_E + r\frac{\partial\Psi_E}{\partial r}\right)\right]. \tag{22}$$

This shows that the projections $\mathbf{r}\cdot\mathbf{E}'$ and $\mathbf{r}\cdot\mathbf{B}$ determine Ψ_M, and Ψ_E, respectively, and consequently determine $\mathbf{E}'$ and $\mathbf{B}$ via Equations (15) and (16).

The projections $\mathbf{r}\cdot\mathbf{E}'$ and $\mathbf{r}\cdot\mathbf{B}$ can be determined everywhere by means of the identities

$$\mathbf{r}\cdot(\nabla^2+k^2)\mathbf{E}' = (\nabla^2+k^2)(\mathbf{r}\cdot\mathbf{E}') \tag{23}$$

and

$$\mathbf{r}\cdot(\nabla^2+k^2)\mathbf{B} = (\nabla^2+k^2)(\mathbf{r}\cdot\mathbf{B}) \tag{24}$$

together with Equations (13) and (14). These together give the inhomogeneous scalar Helmholtz equations

$$(\nabla^2+k^2)(\mathbf{r}\cdot\mathbf{E}') = \mathbf{r}\cdot\mathbf{S}_E = \rho_E \tag{25}$$

and

$$(\nabla^2+k^2)(\mathbf{r}\cdot\mathbf{B}) = \mathbf{r}\cdot\mathbf{S}_B = \rho_B \tag{26}$$

where ρ_E and ρ_B are given by

$$\rho_E = -i\omega\mu\mathbf{r}\cdot\left(\frac{\nabla\times\nabla\times\mathbf{J}_S}{k^2} + \nabla\times\mathbf{M}_S\right) \tag{27}$$

and

$$\rho_B = -\mu\mathbf{r}\cdot(\nabla\times\nabla\times\mathbf{M}_S + \nabla\times\mathbf{J}_S). \tag{28}$$

ANGULAR MOMENTUM OPERATOR AND VECTOR SPHERICAL HARMONICS

The subsequent introduction of spherical boundaries near a localized source distribution is greatly facilitated by the introduction of the angular momentum operator

$$\mathbf{L} = -i\mathbf{r}\times\nabla \tag{29}$$

and the normalized vector spherical harmonics

$$\mathbf{X}_{l,m}(\theta,\phi) = \frac{1}{\sqrt{l(l+1)}}\mathbf{L}Y_{l,m}(\theta,\phi). \tag{30}$$

By simple manipulation using the identity $\nabla \times (\mathbf{r}\Psi) = -\mathbf{r} \times \nabla\Psi + \Psi\nabla \times \mathbf{r} = -\mathbf{r} \times \nabla\Psi$, Equations (15) and (16) can be written

$$\mathbf{E}' = -i\mathbf{L}\Psi_E + \frac{\omega}{k^2}\nabla \times \mathbf{L}\Psi_M \tag{31}$$

and

$$\mathbf{B} = -i\mathbf{L}\Psi_M - \frac{1}{\omega}\nabla \times \mathbf{L}\Psi_E. \tag{32}$$

Since Ψ_M and Ψ_E are solutions to the scalar Helmholtz equation, they can be expanded in spherical coordinate eigenfunctions of the form

$$\Psi_{l,m} = f_l(kr)Y_{l,m}(\theta,\phi) \tag{33}$$

where f_l is a spherical Bessel function satisfying the differential equation

$$\frac{d^2 f_l}{dr^2} + \frac{2}{r}\frac{df_l}{dr} + \left(k^2 - \frac{l(l+1)}{r^2}\right)f_l = 0 \tag{34}$$

and the $Y_{l,m}$ are spherical harmonics satisfying, in particular,

$$\mathbf{L}^2 Y_{l,m} = l(l+1)Y_{l,m}. \tag{35}$$

Now note the following series of substitutions resulting in a very useful identity:

$$\begin{aligned}\mathbf{L}[f_l(kr)Y_{l,m}(\theta,\phi)] &= -i\mathbf{r} \times \nabla[f_l(kr)Y_{l,m}(\theta,\phi)] \\ &= -i\mathbf{r} \times [Y_{l,m}(\theta,\phi)\nabla_r f_l(kr) + f_l(kr)\nabla_{\theta,\phi}Y_{l,m}(\theta,\phi)] \\ &= -i\mathbf{r} \times [f_l(kr)\nabla_{\theta,\phi}Y_{l,m}(\theta,\phi)] = f_l(kr)[-i\mathbf{r} \times \nabla Y_{l,m}(\theta,\phi)] \\ &= f_l\mathbf{L}Y_{l,m}(\theta,\phi) = \sqrt{l(l+1)}\,f_l(kr)\mathbf{X}_{l,m}(\theta,\phi).\end{aligned} \tag{36}$$

If the scalar functions are written as the expansions

$$\Psi_E = i\sum_{l=0}^{\infty}\sum_{m=-l}^{m=l}\frac{a^E_{l,m}}{\sqrt{l(l+1)}}f_l(kr)Y_{l,m}(\theta,\phi) \tag{37}$$

and

$$\Psi_M = i\sum_{l=0}^{\infty}\sum_{m=-l}^{m=l}\frac{a^M_{l,m}}{\sqrt{l(l+1)}}g_l(kr)Y_{l,m}(\theta,\phi) \tag{38}$$

then these may be substituted into Equations (31), (32), and (36) to give

$$\mathbf{E}' = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left\{ a_{l,m}^{E} f_l \mathbf{X}_{l,m} + \frac{i\omega}{k^2} a_{l,m}^{M} \nabla \times [g_l \mathbf{X}_{l,m}] \right\} \tag{39}$$

and

$$\mathbf{B} = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left\{ a_{l,m}^{M} g_l \mathbf{X}_{l,m} - \frac{i}{\omega} a_{l,m}^{E} \nabla \times [f_l \mathbf{X}_{l,m}] \right\}. \tag{40}$$

These forms will be used outside of the source region for both the primary fields due to the source, and the scattered fields due to spherical surfaces. The primary fields will be developed first. Then, in a subsequent section, the scattered fields will be developed by applying the appropriate boundary conditions, and utilizing various orthogonality properties of the vector spherical harmonics.

GREEN FUNCTION FOR THE PRIMARY FIELDS

For a localized source and no boundaries, the solutions to Equations (25) and (26) can be expressed in terms of a Green function by means of Green's theorem:

$$\int d^3\mathbf{r}' \{ G(\mathbf{r}-\mathbf{r}') (\nabla'^2 + k^2) F(\mathbf{r}') - F(\mathbf{r}') (\nabla'^2 + k^2) G(\mathbf{r}-\mathbf{r}') \} = 0. \tag{41}$$

With G and F satisfying

$$(\nabla'^2 + k^2) G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}') \tag{42}$$

and

$$(\nabla'^2 + k^2) F(\mathbf{r}') = \rho(\mathbf{r}') \tag{43}$$

Equation (41) reduces to

$$F(\mathbf{r}) = \int d^3\mathbf{r}' G(\mathbf{r}-\mathbf{r}') \rho(\mathbf{r}'). \tag{44}$$

As shown in a number of texts, such as Jackson,[7] the free field Green function is given by

$$G(\mathbf{r}-\mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi |\mathbf{r}-\mathbf{r}'|} \tag{45}$$

and has the expansion in spherical coordinate eigenfunctions given by

$$G(\mathbf{r}-\mathbf{r}') = ik \sum_{l=0}^{\infty} j_l(kr^<) h_l^{(1)}(kr^>) \sum_{m=-l}^{m=l} Y_{l,m}^{*}(\theta', \phi') Y_{l,m}(\theta, \phi) \tag{46}$$

where j_l and $h_l^{(1)}$ are spherical Bessel functions of the first and third kind, respectively, and

$$r^{</>} = r, r </> r', r' \text{ otherwise.} \tag{47}$$

EXPANSION COEFFICIENTS FOR THE PRIMARY FIELDS

The primary fields outside the source region have the forms given in Equations (39) and (40). The expansion coefficients can be determined by evaluating $\mathbf{r} \cdot \mathbf{E}'$ and $\mathbf{r} \cdot \mathbf{B}$. First note that

$$\mathbf{r} \cdot \mathbf{X}_{l,m} = 0 \tag{48}$$

and

$$\mathbf{r} \cdot \nabla f(r) \times \mathbf{X}_{l,m} = \mathbf{X}_{l,m} \cdot \mathbf{r} \times \nabla f(r) = 0. \tag{49}$$

This leads to the identity

$$\mathbf{r} \cdot \nabla \times [f_l(kr)\mathbf{X}_{l,m}] = f_l(kr)\mathbf{r} \cdot \nabla \times \mathbf{X}_{l,m} = f_l(kr)\mathbf{r} \times \nabla \cdot \mathbf{X}_{l,m} \tag{50}$$

$$= if_l(kr)\mathbf{L} \cdot \mathbf{X}_{l,m} = \frac{if_l(kr)}{\sqrt{l(l+1)}}\mathbf{L}^2 Y_{l,m} = if_l(kr)\sqrt{l(l+1)}\,Y_{l,m}.$$

Thus the primary field expansion coefficients are given simply by

$$\mathbf{r} \cdot \mathbf{E}'^P = -\frac{\omega}{k^2}\sum_{l=0}^{\infty}\sqrt{l(l+1)}\,g_l(kr)\sum_{m=-l}^{m=l} a_{l,m}^{M,P} Y_{l,m} \tag{51}$$

and

$$\mathbf{r} \cdot \mathbf{B}^P = \frac{1}{\omega}\sum_{l=0}^{\infty}\sqrt{l(l+1)}\,f_l(kr)\sum_{m=-l}^{m=l} a_{l,m}^{E,P} Y_{l,m}. \tag{52}$$

When Equations (43) and (44) are applied to Equations (25) and (26), and the expansion Equation (46) is used, expansions are obtained for $\mathbf{r} \cdot \mathbf{E}'^P$ and $\mathbf{r} \cdot \mathbf{B}^P$. Then, Equation (51) becomes

$$-\frac{\omega}{k^2}\sum_{l=0}^{\infty}\sqrt{l(l+1)}\,g_l(kr)\sum_{m=-l}^{m=l} a_{l,m}^{M,P} Y_{l,m}(\theta,\phi) \tag{53}$$

$$= ik\sum_{l=0}^{\infty}\sum_{m=-l}^{m=l} Y_{l,m}(\theta,\phi)\int d^3\mathbf{r}'\rho_E(\mathbf{r}')j_l(kr^{<})h_l^{(1)}(kr^{>})Y_{l,m}^{*}(\theta',\phi').$$

For $r < r'$ for any source point, $f_l(kr) = g_l(kr) = j_l(kr)$ and this can be used together with the orthonormality of the spherical harmonics $Y_{l,m}$ to convert Equation (53) to

$$a_{l,m}^{M,P} = -\frac{ik^3}{\omega\sqrt{l(l+1)}}\int d^3\mathbf{r}'\rho_E(\mathbf{r}')h_l^{(1)}(kr')Y_{l,m}^{*}(\theta',\phi'). \tag{54}$$

A similiar treatment of $\mathbf{r} \cdot \mathbf{B}^P$ gives

$$a_{l,m}^{E,P} = \frac{ik\omega}{\sqrt{l(l+1)}} \int d^3\mathbf{r}' \rho_B(\mathbf{r}') h_l^{(1)}(kr') Y_{l,m}^*(\theta', \phi'). \tag{55}$$

For points $r > r'$ for any source point, $h_l^{(1)}(kr')$ is replaced by $j_l(kr')$ in Equations (54) and (55).

APPLICATION TO A SPHERE WITH AN EXTERNAL SOURCE

For points interior to the sphere and exterior points outside the source, the fields have expansions of the forms given in Equations (39) and (40). In particular, these forms hold for all points on the surface of the sphere, and will be useful in imposing boundary conditions. The interior fields will be designated with a 1 superscript, and the exterior scattered fields have a 2 superscript. To be regular at the origin, the interior fields have the form

$$\mathbf{E}^1 = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left\{ a_{l,m}^{E,1} j_l(k_1 r) \mathbf{X}_{l,m} + \frac{i\omega}{k_1^2} a_{l,m}^{M,1} \nabla \times [j_l(k_1 r) \mathbf{X}_{l,m}] \right\} \tag{56}$$

and

$$\mathbf{B}^1 = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left\{ a_{l,m}^{M,1} j_l(k_1 r) \mathbf{X}_{l,m} - \frac{i}{\omega} a_{l,m}^{E,1} \nabla \times [j_l(k_1 r) \mathbf{X}_{l,m}] \right\} \tag{57}$$

while the exterior scattered fields are expressed in terms of outgoing waves and have the form

$$\mathbf{E}^2 = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left\{ a_{l,m}^{E,2} h_l^{(1)}(k_2 r) \mathbf{X}_{l,m} + \frac{i\omega}{k_2^2} a_{l,m}^{M,2} \nabla \times [h_l^{(1)}(k_2 r) \mathbf{X}_{l,m}] \right\} \tag{58}$$

and

$$\mathbf{B}^2 = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left\{ a_{l,m}^{M,2} h_l^{(1)}(k_2 r) \mathbf{X}_{l,m} - \frac{i}{\omega} a_{l,m}^{E,2} \nabla \times [h_l^{(1)}(k_2 r) \mathbf{X}_{l,m}] \right\} \tag{59}$$

and, just outside the sphere surface, the primary fields have the form

$$\mathbf{E}^P = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left\{ a_{l,m}^{E,P} j_l(k_2 r) \mathbf{X}_{l,m} + \frac{i\omega}{k_2^2} a_{l,m}^{M,P} \nabla \times [j_l(k_2 r) \mathbf{X}_{l,m}] \right\} \tag{60}$$

and

$$\mathbf{B}^P = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left\{ a_{l,m}^{M,P} j_l(k_2 r) \mathbf{X}_{l,m} - \frac{i}{\omega} a_{l,m}^{E,P} \nabla \times [j_l(k_2 r) \mathbf{X}_{l,m}] \right\}. \tag{61}$$

BOUNDARY CONDITIONS

The boundary conditions are continuity of the tangential **E** and tangential **H** fields at the surface of the sphere. If R is the radius of the sphere, then the boundary conditions can be expressed as

$$\mathbf{R} \times \mathbf{E}^1 = \mathbf{R} \times (\mathbf{E}^2 + \mathbf{E}^P) \tag{62}$$

and

$$\frac{1}{\mu_1}\mathbf{R} \times \mathbf{B}^1 = \frac{1}{\mu_2}\mathbf{R} \times (\mathbf{B}^2 + \mathbf{B}^P). \tag{63}$$

To apply the boundary conditions in Equations (62) and (63) it is necessary to develop a rather complex identity. First, note that

$$\mathbf{r} \times \nabla \times [f(r)\mathbf{X}_{l,m}] = \mathbf{r} \times \left[\frac{\dot{f}(r)}{r}\mathbf{r} \times \mathbf{X}_{l,m} + f(r)\nabla \times \mathbf{X}_{l,m} \right]. \tag{64}$$

The first term on the right-hand side of Equation (64) has the form

$$\mathbf{r} \cdot \mathbf{X}_{l,m}\frac{\dot{f}(r)}{r}\mathbf{r} - r\dot{f}(r)\mathbf{X}_{l,m} = -r\dot{f}(r)\mathbf{X}_{l,m} \tag{65}$$

that is, it is proportional to $\mathbf{X}_{l,m}$. Next, it will be demonstrated that the second term on the right-hand side of Equation (64) also is proportional to $\mathbf{X}_{l,m}$.

First, note that by a standard identity

$$\nabla \times (\mathbf{r} \times \nabla)F = \mathbf{r}\nabla^2 F - (\nabla \cdot \mathbf{r})\nabla F + (\nabla F \cdot \nabla)\mathbf{r} - (\mathbf{r} \cdot \nabla)\nabla F. \tag{66}$$

The second term on the right-hand side of Equation (66) is just $-3\nabla F$. For the third term on the right-hand side of Equation (66)

$$(\nabla F \cdot \nabla)\mathbf{r} = \partial_i F \partial_i r_j = \partial_i F \delta_{i,j} = \partial_j F = \nabla F \tag{67}$$

and, the fourth term on the right-hand side of Equation (66) is

$$-(\mathbf{r} \cdot \nabla)\nabla F = -r_i\partial_i\partial_j F = -r_i\partial_j\partial_i F \tag{68}$$

$$= -(\partial_j r_i - \delta_{i,j})\partial_i F = \nabla F - \nabla(\mathbf{r} \cdot \nabla F).$$

Applying these results to Equation (66) produces the identity

$$\nabla \times (\mathbf{r} \times \nabla)F = \mathbf{r}\nabla^2 F - \nabla\left(F + r\frac{\partial F}{\partial r}\right). \tag{69}$$

This result can be applied directly to $\nabla \times \mathbf{X}_{l,m}$:

$$\nabla \times \mathbf{X}_{l,m} = \frac{1}{\sqrt{l(l+1)}} \nabla \times \mathbf{L} Y_{l,m} = -\frac{i}{\sqrt{l(l+1)}} \nabla \times (\mathbf{r} \times \nabla) Y_{l,m} \tag{70}$$

$$= -\frac{i}{\sqrt{l(l+1)}} \left[\mathbf{r} \nabla^2 Y_{l,m} - \nabla \left(1 + r \frac{\partial}{\partial r} \right) Y_{l,m} \right].$$

but,

$$\nabla^2 Y_{l,m} = -\frac{\mathbf{L}^2}{r^2} Y_{l,m} = -\frac{l(l+1)}{r^2} Y_{l,m} \tag{71}$$

so,

$$\nabla \times \mathbf{X}_{l,m} = \frac{i l(l+1)}{r^2} \mathbf{r} Y_{l,m} + \frac{i}{\sqrt{l(l+1)}} \nabla Y_{l,m} \tag{72}$$

and

$$\mathbf{r} \times \nabla \times \mathbf{X}_{l,m} = \frac{i}{\sqrt{l(l+1)}} \mathbf{r} \times \nabla Y_{l,m} = -\mathbf{X}_{l,m}. \tag{73}$$

With all the foregoing, Equation (64) reduces (for spherical Bessel functions $f_l(kr)$) to

$$\mathbf{r} \times \nabla \times [f_l(kr) \mathbf{X}_{l,m}] = -[f_l(kr) + kr \dot{f}_l(kr)] \mathbf{X}_{l,m}. \tag{74}$$

Application of the boundary conditions Equations (62) and (63), using the expansions Equations (56) through (61), and the identity Equation (74) gives

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \{ \left[\frac{a_{l,m}^{M,1} j_l(k_1 R)}{\mu_1} - \frac{a_{l,m}^{M,2} h_l^{(1)}(k_2 R)}{\mu_2} - \frac{a_{l,m}^{M,P} j_l(k_2 R)}{\mu_2} \right] \mathbf{R} \times \mathbf{X}_{l,m} \tag{75}$$

$$+ i [\frac{a_{l,m}^{E,1} \{ j_l(k_1 R) + k_1 R \dot{j}_l(k_1 R) \}}{\omega \mu_1} - \frac{a_{l,m}^{E,2} \{ h_l^{(1)}(k_2 R) + k_2 R \dot{h}_l^{(1)}(k_2 R) \}}{\omega \mu_2}$$

$$- \frac{a_{l,m}^{E,P} \{ j_l(k_2 R) + k_2 R \dot{j}_l(k_2 R) \}}{\omega \mu_2}] \mathbf{X}_{l,m} \} = 0$$

and

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{m=l}\{[a_{l,m}^{E,1}j_l(k_1R)-a_{l,m}^{E,2}h_l^{(1)}(k_2R)-a_{l,m}^{E,P}j_l(k_2R)]\mathbf{R}\times\mathbf{X}_{l,m} \tag{76}$$

$$-i\omega[\frac{a_{l,m}^{M,1}\{j_l(k_1R)+k_1R\dot{j}_l(k_1R)\}}{k_1^2}-\frac{a_{l,m}^{M,2}\{h_l^{(1)}(k_2R)+k_2R\dot{h}_l^{(1)}(k_2R)\}}{k_2^2}$$

$$-\frac{a_{l,m}^{M,P}\{j_l(k_2R)+k_2R\dot{j}_l(k_2R)\}}{k_2^2}]\mathbf{X}_{l,m}\}=0.$$

The vector spherical harmonics satisfy several orthogonality and normalization conditions which are useful in determining the expansion coefficients[6]

$$\mathbf{X}_{l',m'}^{*}\cdot\mathbf{r}\times\mathbf{X}_{l,m}=0 \tag{77}$$

$$\int d\Omega\mathbf{X}_{l',m'}^{*}\cdot\mathbf{X}_{l,m}=\delta_{l',l}\delta_{m',m} \tag{78}$$

and

$$\int d\Omega(\mathbf{r}\times\mathbf{X}_{l',m'}^{*})\cdot(\mathbf{r}\times\mathbf{X}_{l,m}) \tag{79}$$

$$=\int d\Omega[(\mathbf{r}\cdot\mathbf{r})(\mathbf{X}_{l',m'}^{*}\cdot\mathbf{X}_{l,m})-(\mathbf{r}\cdot\mathbf{X}_{l',m'}^{*})(\mathbf{r}\cdot\mathbf{X}_{l,m})]$$

$$=r^2\delta_{l',l}\delta_{m',m}$$

where the integration is over all solid angles, and $\delta_{i,j}$ is the Kronecker delta. If Equations (75) and (76) are scalar multiplied by $\mathbf{X}_{l',m'}^{*}$ or $\mathbf{R}\times\mathbf{X}_{l',m'}^{*}$ and the solid angle integral performed, Equation (77) through (79) may be applied to give

$$a_{l,m}^{M,1}j_l(k_1R)-\tau a_{l,m}^{M,2}h_l^{(1)}(k_2R)=\tau a_{l,m}^{M,P}j_l(k_2R) \tag{80}$$

$$a_{l,m}^{M,1}[j_l(k_1R)+k_1R\dot{j}_l(k_1R)]-\gamma a_{l,m}^{M,2}[h_l^{(1)}(k_2R)+k_2R\dot{h}_l^{(1)}(k_2R)] \tag{81}$$

$$=\gamma a_{l,m}^{M,P}[j_l(k_2R)+k_2R\dot{j}_l(k_2R)]$$

$$a_{l,m}^{E,1}j_l(k_1R)-a_{l,m}^{E,2}h_l^{(1)}(k_2R)=a_{l,m}^{E,P}j_l(k_2R) \tag{82}$$

and

$$a_{l,m}^{E,1}[j_l(k_1R)+k_1R\dot{j}_l(k_1R)]-\tau a_{l,m}^{E,2}[h_l^{(1)}(k_2R)+k_2R\dot{h}_l^{(1)}(k_2R)] \tag{83}$$

$$=\tau a_{l,m}^{E,P}[j_l(k_2R)+k_2R\dot{j}_l(k_2R)]$$

where $\tau=\mu_1/\mu_2$ and $\gamma=k_1^2/k_2^2$.

Using the Wronskian $\{j_l(u), h_l^{(1)}(u)\} = i/u^2$, introduce the determinants of coefficients

$$D_{E,l} = h_l^{(1)}(u_2)\,[j_l(u_1) + u_1\dot{j}_l(u_1)] - \tau j_l(u_1)\,[h_l^{(1)}(u_2) + u_2\dot{h}_l^{(1)}(u_2)] \tag{84}$$

and

$$D_{M,l} = \tau h_l^{(1)}(u_2)\,[j_l(u_1) + u_1\dot{j}_l(u_1)] - \gamma j_l(u_1)\,[h_l^{(1)}(u_2) + u_2\dot{h}_l^{(1)}(u_2)] \tag{85}$$

and the expressions

$$N_{E,l} = -j_l(u_2)\,[j_l(u_1) + u_1\dot{j}_l(u_1)] + \tau j_l(u_1)\,[j_l(u_2) + u_2\dot{j}_l(u_2)] \tag{86}$$

and

$$N_{M,l} = -\tau j_l(u_2)\,[j_l(u_1) + u_1\dot{j}_l(u_1)] + \gamma j_l(u_1)\,[j_l(u_2) + u_2\dot{j}_l(u_2)]. \tag{87}$$

where $u_1 = k_1 R$ and $u_2 = k_2 R$. Then, the solutions for the expansion coefficients, *valid for any localized exterior source distribution that does not include the sphere surface*, are given by

$$a_{l,m}^{E,1} = -\frac{i\tau}{u_2 D_{E,l}} a_{l,m}^{E,P} \tag{88}$$

$$a_{l,m}^{E,2} = \frac{N_{E,l}}{D_{E,l}} a_{l,m}^{E,P} \tag{89}$$

$$a_{l,m}^{M,1} = -\frac{i\tau\gamma}{u_2 D_{M,l}} a_{l,m}^{M,P} \tag{90}$$

and

$$a_{l,m}^{M,2} = \frac{N_{M,l}}{D_{M,l}} a_{l,m}^{M,P}. \tag{91}$$

ELECTRIC AND MAGNETIC FIELDS FOR A PERFECTLY CONDUCTING SPHERE

For a highly conducting sphere, the numerical results using the coefficient representations in Equations (84) through (91) may be unreliable. In this case it is best to treat the sphere as a perfect conductor. Then, the coefficients may be determined by means of the single boundary condition Equation (62)

$$\mathbf{R} \times (\mathbf{E} + \mathbf{E}^P) = 0 \tag{92}$$

at the surface of the sphere. Applying the expansions Equations (58) and (60), and the identity Equation (74), and dropping the 2 subscript/superscript for the exterior region, Equation (92) becomes

$$\sum_{l=0}^{\infty}\sum_{-m}^{m}[-\frac{i\omega}{k^2}\{a_{l,m}^{M}[h_l^{(1)}(kR)+kR\dot{h}_l^{(1)}(kR)]+a_{l,m}^{M,P}[j_l(kR)+kR\dot{j}_l(kR)]\}\mathbf{X}_{l,m} \tag{93}$$

$$\{a_{l,m}^{E}h_l^{(1)}(kR)+a_{l,m}^{E,P}j_l(kR)\}\mathbf{R}\times\mathbf{X}_{l,m}]=0.$$

Again, the properties of the vector spherical harmonics are used in Equations (77) through (79), and the resulting equations solved to give

$$a_{l,m}^{M}=\frac{N_{M,l}}{D_{M,l}}a_{l,m}^{M,P} \tag{94}$$

and

$$a_{l,m}^{E}=\frac{N_{E,l}}{D_{E,l}}a_{l,m}^{E,P} \tag{95}$$

again *valid for any localized **exterior** source distribution not containing the sphere surface*, where now,

$$N_{M,l}=-[j_l(kR)+kR\dot{j}_l(kR)] \tag{96}$$

$$D_{M,l}=h_l^{(1)}(kR)+kR\dot{h}_l^{(1)}(kR) \tag{97}$$

$$N_{E,l}=-j_l(kR) \tag{98}$$

and

$$D_{E,l}=h_l^{(1)}(kR). \tag{99}$$

SPECIALIZATION TO DIPOLE SOURCES

Explicit solutions will be constructed for current and magnetic dipole sources. For these, the current density and magnetization take the form

$$\mathbf{J}_S(\mathbf{r})=\mathbf{p}\delta(\mathbf{r}-\mathbf{R}_0) \tag{100}$$

and

$$\mathbf{M}_S(\mathbf{r})=\mathbf{m}\delta(\mathbf{r}-\mathbf{R}_0). \tag{101}$$

The current dipole moment $\mathbf{p}$ can be specified directly for a dipole source shorted to the medium, and has a non-zero value at dc. To represent an insulated dipole that couples only through the displacement current, $\mathbf{p}$ should be written $\mathbf{p}=i\omega\mathbf{p}'$, where $\mathbf{p}'$ is the electric dipole moment.

For $\mathbf{V}(\mathbf{r}')=\mathbf{v}\delta(\mathbf{r}'-\mathbf{R}_0)$, with $\mathbf{v}$ a constant vector, the following identities are valid:

$$\nabla' \times \mathbf{V} = -\mathbf{v} \times \nabla'\delta(\mathbf{r}' - \mathbf{R}_0) \tag{102}$$

$$\nabla' \times \nabla' \times \mathbf{V} = -\mathbf{v}\nabla'^2\delta(\mathbf{r}' - \mathbf{R}_0) + (\mathbf{v} \cdot \nabla')\nabla'\delta(\mathbf{r}' - \mathbf{R}_0) \tag{103}$$

$$\int d\mathbf{r}'^3 F(\mathbf{r}, \mathbf{r}')\nabla'\delta(\mathbf{r}' - \mathbf{R}_0) = -\nabla_0 F(\mathbf{r}, \mathbf{R}_0) \tag{104}$$

$$\int d\mathbf{r}'^3 F(\mathbf{r}, \mathbf{r}')\nabla'^2\delta(\mathbf{r}' - \mathbf{R}_0) = \nabla_0^2 F(\mathbf{r}, \mathbf{R}_0) \tag{105}$$

and

$$\int d\mathbf{r}'^3 F(\mathbf{r}, \mathbf{r}')\nabla' \otimes \nabla'\delta(\mathbf{r}' - \mathbf{R}_0) = \nabla_0 \otimes \nabla_0 F(\mathbf{r}, \mathbf{R}_0). \tag{106}$$

If Equations (100) and (101) are substituted in Equations (54) and (55) and Equations (102) through (106) applied, the resulting source expansion coefficients, separated according to source type, are (again, for an exterior source distribution, $r < r'$ for any source point near enough to the sphere surface)

$$a_{l,m}^{E,CD} = \frac{k_2^3\mu_2}{\omega\varepsilon_2\sqrt{l(l+1)}}(\mathbf{R}_0 \times \mathbf{p}) \cdot \nabla_0[h_l^{(1)}(k_2R_0)Y_{l,m}^*(\theta_0,\phi_0)] \tag{107}$$

$$a_{l,m}^{M,CD} = -\frac{k_2^3\mu_2}{\omega^2\varepsilon_2\sqrt{l(l+1)}}\{k_2^2(\mathbf{R}_0 \cdot \mathbf{p})h_l^{(1)}(k_2R_0)Y_{l,m}^*(\theta_0,\phi_0) \tag{108}$$

$$+\mathbf{p} \cdot \nabla_0\left[Y_{l,m}^*(\theta_0,\phi_0)\frac{d}{dR_0}\{R_0h_l^{(1)}(k_2R_0)\}\right]\}$$

$$a_{l,m}^{E,MD} = \frac{k_2\omega\mu_2}{\sqrt{l(l+1)}}\{k_2^2(\mathbf{R}_0 \cdot \mathbf{m})h_l^{(1)}(k_2R_0)Y_{l,m}^*(\theta_0,\phi_0) \tag{109}$$

$$+\mathbf{m} \cdot \nabla_0\left[Y_{l,m}^*(\theta_0,\phi_0)\frac{d}{dR_0}\{R_0h_l^{(1)}(k_2R_0)\}\right]\}$$

and

$$a_{l,m}^{M,MD} = -\frac{k_2^3\mu_2}{\sqrt{l(l+1)}}(\mathbf{R}_0 \times \mathbf{m}) \cdot \nabla_0[h_l^{(1)}(k_2R_0)Y_{l,m}^*(\theta_0,\phi_0)]. \tag{110}$$

There are two cases for both types of dipole, radial and tangential. For the radial case

$$a_{l,m}^{E,CD,R} = 0 \tag{111}$$

$$a_{l,m}^{M,CD,R} = -\frac{pR_0k_2^3\mu_2}{\omega^2\varepsilon_2\sqrt{l(l+1)}}\left[k_2^2h_l^{(1)}(k_2R_0)+\frac{1}{R_0}\frac{d^2}{dR_0^2}\{R_0h_l^{(1)}(k_2R_0)\}\right]Y_{l,m}^*(\theta_0,\phi_0) \tag{112}$$

$$a_{l,m}^{E,MD,R} = \frac{\mathrm{m}\omega R_0k_2\mu_2}{\sqrt{l(l+1)}}\left[k_2^2h_l^{(1)}(k_2R_0)+\frac{1}{R_0}\frac{d^2}{dR_0^2}\{R_0h_l^{(1)}(k_2R_0)\}\right]Y_{l,m}^*(\theta_0,\phi_0) \tag{113}$$

and

$$a_{l,m}^{M,MD,R} = 0. \tag{114}$$

Without loss of generality, the tangential dipole can be taken to have only a θ component. Then Equations (107) through (110) become

$$a_{l,m}^{E,CD,T} = \frac{pk_2^3\mu_2}{\omega\varepsilon_2\sqrt{l(l+1)}}h_l^{(1)}(k_2R_0)\frac{1}{\sin\theta_0}\frac{\partial}{\partial\phi_0}Y_{l,m}^*(\theta_0,\phi_0) \tag{115}$$

$$a_{l,m}^{M,CD,T} = -\frac{pk_2^3\mu_2}{\omega^2\varepsilon_2\sqrt{l(l+1)}}\frac{1}{R_0}\frac{d}{dR_0}[R_0h_l^{(1)}(k_2R_0)]\frac{\partial}{\partial\theta_0}Y_{l,m}^*(\theta_0,\phi_0) \tag{116}$$

$$a_{l,m}^{E,MD,T} = \frac{\mathrm{m}k_2\omega\mu_2}{\sqrt{l(l+1)}}\frac{1}{R_0}\frac{d}{dR_0}[R_0h_l^{(1)}(k_2R_0)]\frac{\partial}{\partial\theta_0}Y_{l,m}^*(\theta_0,\phi_0) \tag{117}$$

and

$$a_{l,m}^{M,MD,T} = -\frac{\mathrm{m}k_2^3\mu_2}{\sqrt{l(l+1)}}h^{1})_l(k_2R_0)\frac{1}{\sin\theta_0}\frac{\partial}{\partial\phi_0}Y_{l,m}^*(\theta_0,\phi_0). \tag{118}$$

To proceed further, it is useful to list some explicit results for the spherical harmonics and related functions:

$$Y_{l,m}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{-im\phi}. \tag{119}$$

$$P_l^m(\cos\theta) \quad \alpha \quad \sin^m\theta \quad , \quad \theta\to 0 \quad , \quad |m|\geq 1 \tag{120}$$

$$P_l^{-m}(\cos\theta) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta) \tag{121}$$

and

$$\frac{d}{d\theta}P_l^m(\cos\theta) = -(l+m)(l-m+1)P_l^{m-1}(\cos\theta)+m\cot\theta P_l^m(\cos\theta). \tag{122}$$

Again without loss of generality, the dipole may be taken to be on the z-axis: $\theta_0 \rightarrow 0^+$. Then, application of Equations (119) and (120) to Equations (112) and (113) shows that the source coefficients for the radial dipole vanish unless $m = 0$, while application of Equations (119) through (122) to Equations (115) through (118) shows that the source coefficients for the tangential dipole vanish unless $m = \pm 1$.

Before writing down the final results for the source coefficients, it is useful to list some special limits:

$$P_l^0(1) = 1 \tag{123}$$

$$\dot{P}_l^0(1) = \frac{l(l+1)}{2} \tag{124}$$

$$\sin\theta \dot{P}_l^1(\cos\theta)|_{\theta \rightarrow 0^+} = \frac{l(l+1)}{2} \tag{125}$$

and

$$\sin\theta \dot{P}_l^{-1}(\cos\theta)|_{\theta \rightarrow 0^+} = -\frac{1}{2}. \tag{126}$$

These can be used to simplify the dipole source coefficients. For points $r < R_0$ the explicit forms are

$$a_{l,0}^{E,CD,R} = 0 \tag{127}$$

$$a_{l,0}^{M,CD,R} = \frac{p k_2 \mu_2}{R_0} \sqrt{\frac{(2l+1)l(l+1)}{4\pi}} h_l^{(1)}(k_2 R_0) \tag{128}$$

$$a_{l,0}^{E,MD,R} = \frac{i \omega m k_2 \mu_2}{R_0} \sqrt{\frac{(2l+1)l(l+1)}{4\pi}} h_l^{(1)}(k_2 R_0) \tag{129}$$

$$a_{l,0}^{M,MD,R} = 0 \tag{130}$$

$$a_{l,-1}^{E,CD,T} = a_{l,1}^{E,CD,T} = -\frac{\omega p k_2 \mu_2}{2} \sqrt{\frac{2l+1}{4\pi}} h_l^{(1)}(k_2 R_0) \tag{131}$$

$$a_{l,\pm 1}^{M,CD,T} = \mp \frac{p k_2 \mu_2}{2R_0} \sqrt{\frac{2l+1}{4\pi}} [h_l^{(1)}(k_2 R_0) + k_2 R_0 \dot{h}_l^{(1)}(k_2 R_0)] \tag{132}$$

$$a_{l,\pm 1}^{E,MD,T} = \mp \frac{i \omega m k_2 \mu_2}{2R_0} \sqrt{\frac{2l+1}{4\pi}} [h_l^{(1)}(k_2 R_0) + k_2 R_0 \dot{h}_l^{(1)}(k_2 R_0)] \tag{133}$$

and

$$a_{l,-1}^{M,MD,T} = a_{l,1}^{M,MD,T} = \frac{imk_2^3\mu_2}{2}\sqrt{\frac{2l+1}{4\pi}}h_l^{(1)}(k_2R_0). \tag{134}$$

For $r > R_0$, $h_l^{(1)}(k_2R_0)$ is replaced by $j_l(k_2R_0)$ in Equations (127) through (134).

EXPRESSIONS FOR THE SCATTERED FIELDS FOR AN EXTERNAL DIPOLE

The scattered fields are determined by combining Equations (127) through (134), Equations (84) through (91) and Equations (56) through (59) with m suitably restricted. In the manipulation of $\mathbf{X}_{l,0}$ and $\mathbf{X}_{l,\pm 1}$ some useful relationships are

$$2\cos\theta\dot{P}_l^0 - \sin^2\theta\ddot{P}_l^0 = l(l+1)P_l^0 \tag{135}$$

and

$$\frac{P_l^1}{\sin^2\theta} + 2\cos\theta\dot{P}_l^1 - \sin^2\theta\ddot{P}_l^1 = l(l+1)P_l^1. \tag{136}$$

These appear in in the construction of the radial components of the fields, and are a consequence of Equation (50).

In the final presentation of the field expressions, only the Legendre polynomials $P_l \equiv P_l^0$ and their first derivatives will appear by virtue of the relations

$$\frac{P_l^1}{\sin\theta} = -\dot{P}_l \quad \text{and} \quad \dot{P}_l^1\sin\theta = l(l+1)P_l - \cos\theta\dot{P}_l \tag{137}$$

and the Bessel function derivative will be eliminated via the relation

$$f_l(u) + u\dot{f}_l(u) = (l+1)f_l(u) - uf_{l+1}(u). \tag{138}$$

MAGNETIC DIPOLE

Radial

(Interior)

$$E_r^1 = 0 \tag{139}$$

$$E_\theta^1 = 0 \tag{140}$$

$$E_\phi^1 = \frac{i\omega m\mu_1 \sin\theta}{4\pi R_0 R} \sum_{l=0}^{\infty} (2l+1) \frac{1}{D_{E,l}} j_l(k_1 r) h_l^{(1)}(k_2 R_0) \dot{P}_l \tag{141}$$

$$B_r^1 = \frac{m\mu_1}{4\pi R_0 R r} \sum_{l=0}^{\infty} \frac{1}{D_{E,l}} (2l+1) l(l+1) j_l(k_1 r) h_l^{(1)}(k_2 R_0) P_l \tag{142}$$

$$B_\theta^1 = -\frac{m\mu_1 \sin\theta}{4\pi R_0 R r} \sum_{l=0}^{\infty} (2l+1) \frac{1}{D_{E,l}} [(l+1) j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] h_l^{(1)}(k_2 R_0) \dot{P}_l \tag{143}$$

$$B_\phi^1 = 0 \tag{144}$$

(Exterior)

$$E_r^2 = 0 \tag{145}$$

$$E_\theta^2 = 0 \tag{146}$$

$$E_\phi^2 = -\frac{\omega m\mu_2 k_2 \sin\theta}{4\pi R_0} \sum_{l=0}^{\infty} (2l+1) \frac{N_{E,l}}{D_{E,l}} h_l^{(1)}(k_2 r) h_l^{(1)}(k_2 R_0) \dot{P}_l \tag{147}$$

$$B_r^2 = \frac{i m\mu_2 k_2}{4\pi R_0 r} \sum_{l=0}^{\infty} (2l+1) l(l+1) \frac{N_{E,l}}{D_{E,l}} h_l^{(1)}(k_2 r) h_l^{(1)}(k_2 K_0) P_l \tag{148}$$

$$B_\theta^2 = -\frac{i m\mu_2 k_2 \sin\theta}{4\pi R_0 r} \sum_{l=0}^{\infty} (2l+1) \frac{N_{E,l}}{D_{E,l}} [(l+1) h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] h_l^{(1)}(k_2 R_0) \dot{P}_l \tag{149}$$

$$B_\phi^2 = 0 \tag{150}$$

Tangential

(Interior)

$$E_r^1 = \frac{i\omega m\mu_1 \sin\theta \sin\phi}{4\pi R r} \sum_{l=1}^{\infty} (2l+1) \frac{1}{D_{M,l}} j_l(k_1 r) h_l^{(1)}(k_2 R_0) \dot{P}_l \tag{151}$$

$$E_\theta^1 = \frac{i\omega m\mu_1 \sin\phi}{4\pi R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \{ \frac{-1}{D_{E,l}} r j_l(k_1 r) [(l+1) h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] \dot{P}_l \tag{152}$$

$$+ \frac{1}{D_{M,l}} [(l+1) j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] R_0 h_l^{(1)}(k_2 R_0) [l(l+1) P_l - \cos\theta \dot{P}_l] \}$$

$$E_\phi^1 = -\frac{i\omega m\mu_1 \cos\phi}{4\pi R_0 R r}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\{\frac{-1}{D_{M,l}}R_0 h_l^{(1)}(k_2R_0)\,[(l+1)j_l(k_1r)-k_1 r j_{l+1}(k_1r)]\dot{P}_l \tag{153}$$

$$+\frac{1}{D_{E,l}}r j_l(k_1r)\,[(l+1)h_l^{(1)}(k_2R_0)-k_2R_0h_{l+1}^{(1)}(k_2R_0)]\,[l(l+1)P_l-\cos\theta\dot{P}_l]\}$$

$$B_r^1 = \frac{m\mu_1 \sin\theta\cos\phi}{4\pi R_0 R r}\sum_{l=1}^{\infty}(2l+1)\frac{1}{D_{E,l}}[(l+1)h_l^{(1)}(k_2R_0)-k_2R_0h_{l+1}^{(1)}(k_2R_0)]j_l(k_1r)\dot{P}_l \tag{154}$$

$$B_\theta^1 = \frac{m\mu_1 \cos\phi}{4\pi R_0 R r}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\{\frac{1}{D_{M,l}}rR_0k_1^2 j_l(k_1r)h_l^{(1)}(k_2R_0)\dot{P}_l \tag{155}$$

$$+\frac{1}{D_{E,l}}[(l+1)j_l(k_1r)-k_1rj_{l+1}(k_1r)]\,[(l+1)h_l^{(1)}(k_2R_0)-k_2R_0h_{l+1}^{(1)}(k_2R_0)]\,[l(l+1)P_l-\cos\theta\dot{P}_l]\}$$

$$B_\phi^1 = -\frac{m\mu_1 \sin\phi}{4\pi R_0 R r}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\{\frac{1}{D_{M,l}}rR_0k_1^2 j_l(k_1r)h_l^{(1)}(k_2R_0)\,[l(l+1)P_l-\cos\theta\dot{P}_l] \tag{156}$$

$$+\frac{1}{D_{E,l}}[(l+1)j_l(k_1r)-k_1rj_{l+1}(k_1r)]\,[(l+1)h_l^{(1)}(k_2R_0)-k_2R_0h_{l+1}^{(1)}(k_2R_0)]\dot{P}_l\}$$

(Exterior)

$$E_r^2 = -\frac{\omega m\mu_2 k_2 \sin\theta\sin\phi}{4\pi r}\sum_{l=1}^{\infty}(2l+1)\frac{N_{M,l}}{D_{M,l}}h_l^{(1)}(k_2r)h_l^{(1)}(k_2R_0)\dot{P}_l \tag{157}$$

$$E_\theta^2 = -\frac{\omega m\mu_2 k_2 \sin\phi}{4\pi R_0 r}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\{-\frac{N_{E,l}}{D_{E,l}}rh_l^{(1)}(k_2r)\,[(l+1)h_l^{(1)}(k_2R_0)-k_2R_0h_{l+1}^{(1)}(k_2R_0)]\dot{P}_l \tag{158}$$

$$+\frac{N_{M,l}}{D_{M,l}}[(l+1)h_l^{(1)}(k_2r)-k_2rh_{l+1}^{(1)}(k_2r)]R_0h_l^{(1)}(k_2R_0)\,[l(l+1)P_l-\cos\theta\dot{P}_l]\}$$

$$E_\phi^2 = \frac{\omega m\mu_2 k_2 \cos\phi}{4\pi R_0 r}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\{-\frac{N_{M,l}}{D_{M,l}}R_0h_l^{(1)}(k_2R_0)\,[(l+1)h_l^{(1)}(k_2r)-k_2rh_{l+1}^{(1)}(k_2r)]\dot{P}_l \tag{159}$$

$$+\frac{N_{E,l}}{D_{E,l}}rh_l^{(1)}(k_2r)\,[(l+1)h_l^{(1)}(k_2R_0)-k_2R_0h_{l+1}^{(1)}(k_2R_0)]\,[l(l+1)P_l-\cos\theta\dot{P}_l]\}$$

$$B_r^2 = \frac{i m\mu_2 k_2 \sin\theta\cos\phi}{4\pi R_0 r}\sum_{l=1}^{\infty}(2l+1)\frac{N_{E,l}}{D_{E,l}}h_l^{(1)}(k_2r)\,[(l+1)h_l^{(1)}(k_2R_0)-k_2R_0h_{l+1}^{(1)}(k_2R_0)]\dot{P}_l \tag{160}$$

$$B_\theta^2 = \frac{im\mu_2 k_2 \cos\phi}{4\pi R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \{ \frac{N_{M,l}}{D_{M,l}} r R_0 k_2^2 h_l^{(1)}(k_2 r) h_l^{(1)}(k_2 R_0) \dot{P}_l \tag{161}$$

$$+ \frac{N_{E,l}}{D_{E,l}} [(l+1) h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)]\, [(l+1) h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)]\, [l(l+1) P_l - \cos\theta \dot{P}_l]\}$$

$$B_\phi^2 = -\frac{im\mu_2 k_2 \sin\phi}{4\pi R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \{ \frac{N_{M,l}}{D_{M,l}} r R_0 k_2^2 h_l^{(1)}(k_2 r) h_l^{(1)}(k_2 R_0)\, [l(l+1) P_l - \cos\theta \dot{P}_l] \tag{162}$$

$$+ \frac{N_{E,l}}{D_{E,l}} [(l+1) h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)]\, [(l+1) h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] \dot{P}_l\}$$

CURRENT DIPOLE

Radial

(Interior)

$$E_r^1 = -\frac{i\omega p \mu_1}{2\pi k_2^2 R_0 R r} \sum_{l=0}^{\infty} \frac{1}{D_{M,l}} (2l+1) l(l+1) j_l(k_1 r) h_l^{(1)}(k_2 R_0) P_l \tag{163}$$

$$E_\theta^1 = \frac{i\omega p \mu_1 \sin\theta}{2\pi k_2^2 R_0 R r} \sum_{l=0}^{\infty} (2l+1) \frac{1}{D_{M,l}} [(l+1) j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] h_l^{(1)}(k_2 R_0) \dot{P}_l \tag{164}$$

$$E_\phi^1 = 0 \tag{165}$$

$$B_r^1 = 0 \tag{166}$$

$$B_\theta^1 = 0 \tag{167}$$

$$B_\phi^1 = -\frac{p \mu_1 \gamma \sin\theta}{2\pi R_0 R} \sum_{l=0}^{\infty} (2l+1) \frac{1}{D_{M,l}} j_l(k_1 r) h_l^{(1)}(k_2 R_0) \dot{P}_l \tag{168}$$

(Exterior)

$$E_r^2 = -\frac{\omega p \mu_2}{4\pi k_2 R_0 r} \sum_{l=0}^{\infty} \frac{N_{M,l}}{D_{M,l}} (2l+1) l(l+1) h_l^{(1)}(k_2 r) h_l^{(1)}(k_2 R_0) P_l \tag{169}$$

$$E_\theta^2 = \frac{\omega p \mu_2 \sin\theta}{4\pi k_2 R_0 r} \sum_{l=0}^{\infty} (2l+1) \frac{N_{M,l}}{D_{M,l}} [(l+1) h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] h_l^{(1)}(k_2 R_0) \dot{P}_l \tag{170}$$

$$E_\phi^2 = 0 \tag{171}$$

$$B_r^2 = 0 \tag{172}$$

$$B_\theta^2 = 0 \tag{173}$$

$$B_\phi^2 = \frac{ip\mu_2 k_2 \sin\theta}{4\pi R_0} \sum_{l=0}^{\infty} (2l+1) \frac{N_{M,l}}{D_{M,l}} h_l^{(1)}(k_2 r) h_l^{(1)}(k_2 R_0) \dot{P}_l \tag{174}$$

<u>**Tangential**</u>

(Interior)

$$E_r^1 = -\frac{i\omega p\mu_1 \sin\theta\cos\phi}{2\pi k_2^2 R_0 R r} \sum_{l=1}^{\infty} (2l+1) \frac{1}{D_{M,l}} j_l(k_1 r) \left[(l+1) h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)\right] \dot{P}_l \tag{175}$$

$$E_\theta^1 = \frac{i\omega p\mu_1 \cos\phi}{4\pi k_2^2 R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \Big\{ \frac{1}{D_{E,l}} r R_0 k_2^2 j_l(k_1 r) h_l^{(1)}(k_2 R_0) \dot{P}_l \tag{176}$$

$$-\frac{2}{D_{M,l}} \left[(l+1) j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)\right] \left[(l+1) h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)\right] \left[l(l+1) P_l - \cos\theta \dot{P}_l\right] \Big\}$$

$$E_\phi^1 = -\frac{i\omega p\mu_1 \sin\phi}{4\pi k_2^2 R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \Big\{ \frac{1}{D_{E,l}} r R_0 k_2^2 j_l(k_1 r) h_l^{(1)}(k_2 R_0) \left[l(l+1) P_l - \cos\theta \dot{P}_l\right] \tag{177}$$

$$-\frac{2}{D_{M,l}} \left[(l+1) j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)\right] \left[(l+1) h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)\right] \dot{P}_l \Big\}$$

$$B_r^1 = \frac{p\mu_1 \sin\theta \sin\phi}{4\pi R r} \sum_{l=1}^{\infty} (2l+1) \frac{1}{D_{E,l}} j_l(k_1 r) h_l^{(1)}(k_2 R_0) \dot{P}_l \tag{178}$$

$$B_\theta^1 = \frac{p\mu_1 \sin\phi}{4\pi R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \Big\{ \frac{2\gamma}{D_{M,l}} r j_l(k_1 r) \left[(l+1) h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)\right] \dot{P}_l \tag{179}$$

$$+\frac{1}{D_{E,l}} \left[(l+1) j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)\right] R_0 h_l^{(1)}(k_2 R_0) \left[l(l+1) P_l - \cos\theta \dot{P}_l\right] \Big\}$$

$$B_\phi^1 = \frac{p\mu_1 \cos\phi}{4\pi R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \Big\{ \frac{1}{D_{E,l}} R_0 h_l^{(1)}(k_2 R_0) \left[(l+1) j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)\right] \dot{P}_l \tag{180}$$

$$+\frac{2\gamma}{D_{M,l}} r j_l(k_1 r) \left[(l+1) h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)\right] \left[l(l+1) P_l - \cos\theta \dot{P}_l\right] \Big\}$$

(Exterior)

$$E_r^2 = -\frac{\omega p \mu_2 \sin\theta\cos\phi}{4\pi k_2 R_0 r}\sum_{l=1}^{\infty}(2l+1)\frac{N_{M,l}}{D_{M,l}}h_l^{(1)}(k_2r)\,[(l+1)h_l^{(1)}(k_2R_0) - k_2R_0h_{l+1}^{(1)}(k_2R_0)]\dot{P}_l \qquad (181)$$

$$E_\theta^2 = -\frac{\omega p \mu_2 \cos\phi}{4\pi k_2 R_0 r}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\{\frac{N_{E,l}}{D_{E,l}}rR_0k_2^2h_l^{(1)}(k_2r)h_l^{(1)}(k_2R_0)\dot{P}_l \qquad (182)$$

$$+\frac{N_{M,l}}{D_{M,l}}[(l+1)h_l^{(1)}(k_2r) - k_2rh_{l+1}^{(1)}(k_2r)]\,[(l+1)h_l^{(1)}(k_2R_0) - k_2R_0h_{l+1}^{(1)}(k_2R_0)]\,[l(l+1)P_l - \cos\theta\dot{P}_l]\}$$

$$E_\phi^2 = \frac{\omega p \mu_2 \sin\phi}{4\pi k_2 R_0 r}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\{\frac{N_{E,l}}{D_{E,l}}rR_0k_2^2h_l^{(1)}(k_2r)h_l^{(1)}(k_2R_0)\,[l(l+1)P_l - \cos\theta\dot{P}_l] \qquad (183)$$

$$+\frac{N_{M,l}}{D_{M,l}}[(l+1)h_l^{(1)}(k_2r) - k_2rh_{l+1}^{(1)}(k_2r)]\,[(l+1)h_l^{(1)}(k_2R_0) - k_2R_0h_{l+1}^{(1)}(k_2R_0)]\dot{P}_l\}$$

$$B_r^2 = \frac{ip\mu_2k_2\sin\theta\sin\phi}{4\pi r}\sum_{l=1}^{\infty}(2l+1)\frac{N_{E,l}}{D_{E,l}}h_l^{(1)}(k_2r)h_l^{(1)}(k_2R_0)\dot{P}_l \qquad (184)$$

$$B_\theta^2 = \frac{ip\mu_2k_2\sin\phi}{4\pi R_0 r}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\{-\frac{N_{M,l}}{D_{M,l}}rh_l^{(1)}(k_2r)\,[(l+1)h_l^{(1)}(k_2R_0) - k_2R_0h_{l+1}^{(1)}(k_2R_0)]\dot{P}_l \qquad (185)$$

$$+\frac{N_{E,l}}{D_{E,l}}R_0h_l^{(1)}(k_2R_0)\,[(l+1)h_l^{(1)}(k_2r) - k_2rh_{l+1}^{(1)}(k_2r)]\,[l(l+1)P_l - \cos\theta\dot{P}_l]\}$$

$$B_\phi^2 = -\frac{ip\mu_2k_2\cos\phi}{4\pi R_0 r}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\{-\frac{N_{E,l}}{D_{E,l}}R_0h_l^{(1)}(k_2R_0)\,[(l+1)h_l^{(1)}(k_2r) - k_2rh_{l+1}^{(1)}(k_2r)]\dot{P}_l \qquad (186)$$

$$+\frac{N_{M,l}}{D_{M,l}}rh_l^{(1)}(k_2r)\,[(l+1)h_l^{(1)}(k_2R_0) - k_2R_0h_{l+1}^{(1)}(k_2R_0)]\,[l(l+1)P_l - \cos\theta\dot{P}_l]\}$$

DC LIMIT OF THE SCATTERED FIELDS FOR AN EXTERNAL DIPOLE

The dc limits of the field expressions given in Equations (139) through (186) are useful both to provide a more transparent form of expression for a consistency check of the expressions, and to provide explicit expressions for dc applications. These limiting forms are given below. Note that, in the dc limit, $\gamma \to \tau\delta$ where $\delta = \sigma_1/\sigma_2$.

DC MAGNETIC DIPOLE

Radial

(Interior)

$$E_r^1 = 0 \tag{187}$$

$$E_\theta^1 = 0 \tag{188}$$

$$E_\phi^1 = 0 \tag{189}$$

$$B_r^1 = \frac{m\mu_1}{4\pi R_0^3}\sum_{l=1}^{\infty}\frac{(2l+1)l(l+1)}{(\tau+1)l+1}\left(\frac{r}{R_0}\right)^{l-1}P_l \tag{190}$$

$$B_\theta^1 = -\frac{m\mu_1\sin\theta}{4\pi R_0^3}\sum_{l=1}^{\infty}\frac{(2l+1)(l+1)}{(\tau+1)l+1}\left(\frac{r}{R_0}\right)^{l-1}\dot{P}_l \tag{191}$$

$$B_\phi^1 = 0 \tag{192}$$

(Exterior)

$$E_r^2 = 0 \tag{193}$$

$$E_\theta^2 = 0 \tag{194}$$

$$E_\phi^2 = 0 \tag{195}$$

$$B_r^2 = \frac{m\mu_2(\tau-1)}{4\pi R_0 rR}\sum_{l=1}^{\infty}\frac{l(l+1)^2}{(\tau+1)l+1}\left(\frac{R}{r}\right)^{l+1}\left(\frac{R}{R_0}\right)^{l+1}P_l \tag{196}$$

$$B_\theta^2 = \frac{m\mu_2(\tau-1)\sin\theta}{4\pi R_0 rR}\sum_{l=1}^{\infty}\frac{l(l+1)}{(\tau+1)l+1}\left(\frac{R}{r}\right)^{l+1}\left(\frac{R}{R_0}\right)^{l+1}\dot{P}_l \tag{197}$$

$$B_\phi^2 = 0 \tag{198}$$

Tangential

(Interior)

$$E_r^1 = 0 \tag{199}$$

$$E_\theta^1 = 0 \tag{200}$$

$$E_\phi^1 = 0 \tag{201}$$

$$B_r^1 = -\frac{m\mu_1 \sin\theta\cos\phi}{4\pi R_0^3}\sum_{l=1}^{\infty}\frac{(2l+1)l}{(\tau+1)l+1}\left(\frac{r}{R_0}\right)^{l-1}\dot{P}_l \tag{202}$$

$$B_\theta^1 = -\frac{m\mu_1 \cos\phi}{4\pi R_0^3}\sum_{l=1}^{\infty}\frac{(2l+1)}{(\tau+1)l+1}\left(\frac{r}{R_0}\right)^{l-1}[l(l+1)P_l - \cos\theta\dot{P}_l] \tag{203}$$

$$B_\phi^1 = \frac{m\mu_1 \sin\phi}{4\pi R_0^3}\sum_{l=1}^{\infty}\frac{(2l+1)}{(\tau+1)l+1}\left(\frac{r}{R_0}\right)^{l-1}\dot{P}_l \tag{204}$$

(Exterior)

$$E_r^2 = 0 \tag{205}$$

$$E_\theta^2 = 0 \tag{206}$$

$$E_\phi^2 = 0 \tag{207}$$

$$B_r^2 = -\frac{m\mu_2(\tau-1)\sin\theta\cos\phi}{4\pi R_0 rR}\sum_{l=1}^{\infty}\frac{l(l+1)}{(\tau+1)l+1}\left(\frac{R}{r}\right)^{l+1}\left(\frac{R}{R_0}\right)^{l+1}\dot{P}_l \tag{208}$$

$$B_\theta^2 = \frac{m\mu_2(\tau-1)\cos\phi}{4\pi R_0 rR}\sum_{l=1}^{\infty}\frac{l}{(\tau+1)l+1}\left(\frac{R}{r}\right)^{l+1}\left(\frac{R}{R_0}\right)^{l+1}[l(l+1)P_l - \cos\theta\dot{P}_l] \tag{209}$$

$$B_\phi^2 = -\frac{m\mu_2(\tau-1)\sin\phi}{4\pi R_0 rR}\sum_{l=1}^{\infty}\frac{l}{(\tau+1)l+1}\left(\frac{R}{r}\right)^{l+1}\left(\frac{R}{R_0}\right)^{l+1}\dot{P}_l \tag{210}$$

DC CURRENT DIPOLE

Radial

(Interior)

$$E_r^1 = -\frac{p\tau}{2\pi\sigma_2 R_0^3}\sum_{l=1}^{\infty}\frac{(2l+1)l(l+1)}{(\delta+1)l+1}\left(\frac{r}{R_0}\right)^{l-1}P_l \tag{211}$$

$$E_\theta^1 = \frac{p\tau\sin\theta}{2\pi\sigma_2 R_0^3}\sum_{l=1}^{\infty}\frac{(2l+1)(l+1)}{(\delta+1)l+1}\left(\frac{r}{R_0}\right)^{l-1}\dot{P}_l \tag{212}$$

$$E_\phi^1 = 0 \tag{213}$$

$$B_r^1 = 0 \tag{214}$$

$$B_\theta^1 = 0 \tag{215}$$

$$B_\phi^1 = -\frac{p\mu_1\tau\delta r\sin\theta}{2\pi R_0^3}\sum_{l=1}^{\infty}\frac{(2l+1)}{(\delta+1)l+1}\left(\frac{r}{R_0}\right)^{l-1}\dot{P}_l \tag{216}$$

(Exterior)

$$E_r^2 = \frac{p(\delta-1)}{4\pi\sigma_2 R_0 rR}\sum_{l=1}^{\infty}\frac{l(l+1)^2}{(\delta+1)l+1}\left(\frac{R}{r}\right)^{l+1}\left(\frac{R}{R_0}\right)^{l+1}P_l \tag{217}$$

$$E_\theta^2 = \frac{p(\delta-1)\sin\theta}{4\pi\sigma_2 R_0 rR}\sum_{l=1}^{\infty}\frac{l(l+1)}{(\delta+1)l+1}\left(\frac{R}{r}\right)^{l+1}\left(\frac{R}{R_0}\right)^{l+1}\dot{P}_l \tag{218}$$

$$E_\phi^2 = 0 \tag{219}$$

$$B_r^2 = 0 \tag{220}$$

$$B_\theta^2 = 0 \tag{221}$$

$$B_\phi^2 = \frac{p(\delta-1)\mu_2\sin\theta}{4\pi R_0 R}\sum_{l=1}^{\infty}\frac{l+1}{(\delta+1)l+1}\left(\frac{R}{r}\right)^{l+1}\left(\frac{R}{R_0}\right)^{l+1}\dot{P}_l \tag{222}$$

Tangential

(Interior)

$$E_r^1 = \frac{p\tau\sin\theta\cos\phi}{2\pi\sigma_2 R_0^3}\sum_{l=1}^{\infty}\frac{(2l+1)l}{(\delta+1)l+1}\left(\frac{r}{R_0}\right)^{l-1}\dot{P}_l \tag{223}$$

$$E_\theta^1 = \frac{p\tau\cos\phi}{2\pi\sigma_2 R_0^3}\sum_{l=1}^{\infty}\frac{(2l+1)}{(\delta+1)l+1}\left(\frac{r}{R_0}\right)^{l-1}[l(l+1)P_l - \cos\theta\dot{P}_l] \tag{224}$$

$$E_\phi^1 = -\frac{p\tau\sin\phi}{2\pi\sigma_2 R_0^3}\sum_{l=1}^{\infty}\frac{2l+1}{(\delta+1)l+1}\left(\frac{r}{R_0}\right)^{l-1}\dot{P}_l \tag{225}$$

$$B_r^1 = \frac{p\mu_1 \sin\theta \sin\phi}{4\pi R_0^2} \sum_{l=1}^{\infty} \frac{2l+1}{(\tau+1)l+1} \left(\frac{r}{R_0}\right)^{l-1} \dot{P}_l \tag{226}$$

$$B_\theta^1 = \frac{p\mu_1 \sin\phi}{4\pi R_0^2} \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} \left\{ \frac{l+1}{(\tau+1)l+1} [l(l+1)P_l - \cos\theta \dot{P}_l] - \frac{2\tau\delta r l}{(\delta+1)l+1} \dot{P}_l \right\} \left(\frac{r}{R_0}\right)^{l-1} \tag{227}$$

$$B_\phi^1 = \frac{p\mu_1 \cos\phi}{4\pi R_0^2} \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} \left\{ \frac{l+1}{(\tau+1)l+1} \dot{P}_l - \frac{2\tau\delta r l}{(\delta+1)l+1} [l(l+1)P_l - \cos\theta \dot{P}_l] \right\} \left(\frac{r}{R_0}\right)^{l-1} \tag{228}$$

(Exterior)

$$E_r^2 = -\frac{p(\delta-1)\sin\theta\cos\phi}{4\pi\sigma_2 R_0 R r} \sum_{l=1}^{\infty} \frac{l(l+1)}{(\delta+1)l+1} \left(\frac{R}{R_0}\right)^{l+1} \left(\frac{R}{r}\right)^{l+1} \dot{P}_l \tag{229}$$

$$E_\theta^2 = \frac{p(\delta-1)\cos\phi}{4\pi\sigma_2 R_0 R r} \sum_{l=1}^{\infty} \frac{l}{(\delta+1)l+1} \left(\frac{R}{R_0}\right)^{l+1} \left(\frac{R}{r}\right)^{l+1} [l(l+1)P_l - \cos\theta \dot{P}_l] \tag{230}$$

$$E_\phi^2 = -\frac{p(\delta-1)\sin\phi}{4\pi\sigma_2 R_0 R r} \sum_{l=1}^{\infty} \frac{l}{(\delta+1)l+1} \left(\frac{R}{R_0}\right)^{l+1} \left(\frac{R}{r}\right)^{l+1} \dot{P}_l \tag{231}$$

$$B_r^2 = \frac{p\mu_2(\tau-1)\sin\theta\sin\phi}{4\pi R r} \sum_{l=1}^{\infty} \frac{l+1}{(\tau+1)l+1} \left(\frac{R}{R_0}\right)^{l+1} \left(\frac{R}{r}\right)^{l+1} \dot{P}_l \tag{232}$$

$$B_\theta^2 = -\frac{p\mu_2 \sin\phi}{4\pi R R_0 r} \sum_{l=1}^{\infty} \left\{ \frac{R_0(\tau-1)}{(\tau+1)l+1} [l(l+1)P_l - \cos\theta \dot{P}_l] - \frac{r(\delta-1)}{(\delta+1)l+1} \dot{P}_l \right\} \left(\frac{R}{R_0}\right)^{l+1} \left(\frac{R}{r}\right)^{l+1} \tag{233}$$

$$B_\phi^2 = \frac{p\mu_2 \cos\phi}{4\pi R R_0 r} \sum_{l=1}^{\infty} \left\{ -\frac{R_0(\tau-1)}{(\tau+1)l+1} \dot{P}_l + \frac{r(\delta-1)}{(\delta+1)l+1} [l(l+1)P_l - \cos\theta \dot{P}_l] \right\} \left(\frac{R}{R_0}\right)^{l+1} \left(\frac{R}{r}\right)^{l+1} \tag{234}$$

APPLICATION TO A SPHERE WITH AN INTERNAL SOURCE

The scattered fields still have the forms given in Equations (56) through (59). The primary field expansion now has the form

$$\mathbf{E}^P = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left\{ a_{l,m}^{E,P} h_l^{(1)}(k_1 r) \mathbf{X}_{l,m} + \frac{i\omega}{k_1^2} a_{l,m}^{M,P} \nabla \times [h_l^{(1)}(k_1 r) \mathbf{X}_{l,m}] \right\} \tag{235}$$

and

$$\mathbf{B}^P = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left\{ a_{l,m}^{M,P} h_l^{(1)}(k_1 r) \mathbf{X}_{l,m} - \frac{i}{\omega} a_{l,m}^{E,P} \nabla \times [h_l^{(1)}(k_1 r) \mathbf{X}_{l,m}] \right\}. \tag{236}$$

The boundary conditions now are given by

$$\mathbf{R}\times(\mathbf{E}^1+\mathbf{E}^P)=\mathbf{R}\times\mathbf{E}^2 \tag{237}$$

and

$$\frac{1}{\mu_1}\mathbf{R}\times(\mathbf{B}^1+\mathbf{B}^P)=\frac{1}{\mu_2}\mathbf{R}\times\mathbf{B}^2. \tag{238}$$

In expanded form Equations (237) and (238) are

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{m=l}\{\left[\frac{a_{l,m}^{M,1}j_l(k_1R)}{\mu_1}+\frac{a_{l,m}^{M,P}h_l^{(1)}(k_1R)}{\mu_1}-\frac{a_{l,m}^{M,2}h_l^{(1)}(k_2R)}{\mu_2}\right]\mathbf{R}\times\mathbf{X}_{l,m} \tag{239}$$

$$+i[\frac{a_{l,m}^{E,1}\{j_l(k_1R)+k_1R\dot{j}_l(k_1R)\}}{\omega\mu_1}+\frac{a_{l,m}^{E,P}\{h_l^{(1)}(k_1R)+k_1R\dot{h}_l^{(1)}(k_1R)\}}{\omega\mu_1}$$

$$-\frac{a_{l,m}^{E,2}\{h_l^{(1)}(k_2R)+k_2R\dot{h}_l^{(1)}(k_2R)\}}{\omega\mu_2}]\mathbf{X}_{l,m}\}=0$$

and

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{m=l}\{[a_{l,m}^{E,1}j_l(k_1R)+a_{l,m}^{E,P}h_l^{(1)}(k_1R)-a_{l,m}^{E,2}h_l^{(1)}(k_2R)]\mathbf{R}\times\mathbf{X}_{l,m} \tag{240}$$

$$-i\omega[\frac{a_{l,m}^{M,1}\{j_l(k_1R)+k_1R\dot{j}_l(k_1R)\}}{k_1^2}+\frac{a_{l,m}^{M,P}\{h_l^{(1)}(k_1R)+k_1R\dot{h}_l^{(1)}(k_1R)\}}{k_1^2}$$

$$-\frac{a_{l,m}^{M,2}\{h_l^{(1)}(k_2R)+k_2R\dot{h}_l^{(1)}(k_2R)\}}{k_2^2}]\mathbf{X}_{l,m}\}=0.$$

These may be reduced using Equations (77) through (79) to give

$$\bar{\tau}a_{l,m}^{M,1}j_l(k_1R)-a_{l,m}^{M,2}h_l^{(1)}(k_2R)=-\bar{\tau}a_{l,m}^{M,P}h_l^{(1)}(k_1R) \tag{241}$$

$$\bar{\gamma}a_{l,m}^{M,1}[j_l(k_1R)+k_1R\dot{j}_l(k_1R)]-a_{l,m}^{M,2}[h_l^{(1)}(k_2R)+k_2R\dot{h}_l^{(1)}(k_2R)] \tag{242}$$

$$=-\bar{\gamma}a_{l,m}^{M,P}[h_l^{(1)}(k_1R)+k_1R\dot{h}_l^{(1)}(k_1R)]$$

$$a_{l,m}^{E,1}j_l(k_1R)-a_{l,m}^{E,2}h_l^{(1)}(k_2R)=-a_{l,m}^{E,P}h_l^{(1)}(k_1R) \tag{243}$$

and

$$\bar{\tau} a_{l,m}^{E,1}[j_l(k_1R) + k_1R\dot{j}_l(k_1R)] - a_{l,m}^{E,2}[h_l^{(1)}(k_2R) + k_2R\dot{h}_l^{(1)}(k_2R)] \tag{244}$$
$$= -\bar{\tau} a_{l,m}^{E,P}[h_l^{(1)}(k_1R) + k_1R\dot{h}_l^{(1)}(k_1R)]$$

where now $\bar{\tau} = \mu_2/\mu_1$ and $\bar{\gamma} = k_2^2/k_1^2$ are the reciprocals of τ and γ. Proceeding as before, introduce

$$\overline{D}_{E,l} = \bar{\tau} h_l^{(1)}(u_2)\,[(l+1)j_l(u_1) - u_1 j_{l+1}(u_1)] - j_l(u_1)\,[(l+1)h_l^{(1)}(u_2) - u_2 h_{l+1}^{(1)}(u_2)] \tag{245}$$

and

$$\overline{D}_{M,l} = \bar{\gamma} h_l^{(1)}(u_2)\,[(l+1)j_l(u_1) - u_1 j_{l+1}(u_1)] - \bar{\tau} j_l(u_1)\,[(l+1)h_{l+1}^{(1)}(u_2) - u_2 h_{l+1}^{(1)}(u_2)] \tag{246}$$

and the expressions

$$\overline{N}_{E,l} = h_l^{(1)}(u_1)\,[(l+1)h_l^{(1)}(u_2) - u_2 h_{l+1}^{(1)}(u_2)] - \bar{\tau} h_l^{(1)}(u_2)\,[(l+1)h_l^{(1)}(u_1) - u_1 h_{l+1}^{(1)}(u_1)] \tag{247}$$

and

$$\overline{N}_{M,l} = \bar{\tau} h_l^{(1)}(u_1)\,[(l+1)h_l^{(1)}(u_2) - u_2 h_{l+1}^{(1)}(u_2)] - \bar{\gamma} h_l^{(1)}(u_2)\,[(l+1)h_l^{(1)}(u_1) - u_1 h_{l+1}^{(1)}(u_1)]. \tag{248}$$

where again, $u_1 = k_1R$ and $u_2 = k_2R$. Then, the solutions for the expansion coefficients, *valid for any localized **interior** source distribution that does not include the sphere surface*, are given by

$$a_{l,m}^{E,1} = \frac{\overline{N}_{E,l}}{\overline{D}_{E,l}} a_{l,m}^{E,P} \tag{249}$$

$$a_{l,m}^{E,2} = -\frac{i\bar{\tau}}{u_1 \overline{D}_{E,l}} a_{l,m}^{E,P} \tag{250}$$

$$a_{l,m}^{M,1} = \frac{\overline{N}_{M,l}}{\overline{D}_{M,l}} a_{l,m}^{M,P}. \tag{251}$$

and

$$a_{l,m}^{M,2} = \frac{i\overline{\tau\gamma}}{u_1 \overline{D}_{M,l}} a_{l,m}^{M,P} \tag{252}$$

For dipole sources, the expressions in Equations (127) through (134) may be used, with $h_l^{(1)}(k_2R_0)$ replaced by $j_l(k_1R_0)$ and then the 2 subscripts replaced by 1. All the angular information contained in $\mathbf{X}_{l,0}$ and $\mathbf{X}_{l,\pm1}$ remains the same.

The end result is that the field expressions for the source interior to the sphere can be obtained directly from Equations (139) through (186) for the exterior source case, by means of simple exchanges and substitutions.

To obtain the new field components for region 1, replace $h_l^{(1)}(\;)$ by $j_l(\;)$ everywhere in the old expressions for region 2, replace γ, N, and D by $\bar{\gamma}$, $\overline{N}$, and $\overline{D}$ in these expressions, and then replace 2 by 1 everywhere it appears explicitly.

To obtain the new field components for region 2, first define $[f_l(u)] = (l+1)f_l(u) - uf_{l+1}(u)$, then replace $j_l(k_1r)$ and $[j_l(k_1r)]$ by $h_l^{(1)}(k_2r)$ and $[h_l^{(1)}(k_2r)]$, $h_l^{(1)}(k_2R_0)$ and $[h_l^{(1)}(k_2R_0)]$ by $j_l(k_1R_0)$ and $[j_l(k_1R_0)]$, all unbarred quantities by barred quantities, change the remaining subscripts from 1 to 2, and reverse the sign of $\overline{D}_{M,l}$. The results of these operations are listed in the following section.

EXPRESSIONS FOR THE SCATTERED FIELDS FOR AN INTERNAL DIPOLE

MAGNETIC DIPOLE

Radial

(Interior)

$$E_r^1 = 0 \tag{253}$$

$$E_\theta^1 = 0 \tag{254}$$

$$E_\phi^1 = -\frac{\omega m \mu_1 k_1 \sin\theta}{4\pi R_0} \sum_{l=0}^{\infty} (2l+1) \frac{\overline{N}_{E,l}}{\overline{D}_{E,l}} j_l(k_1r) j_l(k_1R_0) \dot{P}_l \tag{255}$$

$$B_r^1 = \frac{i m \mu_1 k_1}{4\pi R_0 r} \sum_{l=0}^{\infty} (2l+1) l(l+1) \frac{\overline{N}_{E,l}}{\overline{D}_{E,l}} j_l(k_1r) j_l(k_1R_0) P_l \tag{256}$$

$$B_\theta^1 = -\frac{i m \mu_1 k_1 \sin\theta}{4\pi R_0 r} \sum_{l=0}^{\infty} (2l+1) \frac{\overline{N}_{E,l}}{\overline{D}_{E,l}} [(l+1) j_l(k_1r) - k_1 r j_{l+1}(k_1r)] j_l(k_1R_0) \dot{P}_l \tag{257}$$

$$B_\phi^1 = 0 \tag{258}$$

(Exterior)

$$E_r^2 = 0 \tag{259}$$

$$E_\theta^2 = 0 \tag{260}$$

$$E_\phi^2 = \frac{i\omega m\mu_2 \sin\theta}{4\pi R_0 R} \sum_{l=0}^{\infty} (2l+1) \frac{1}{\overline{\overline{D}}_{E,l}} h_l^{(1)}(k_2 r) j_l(k_1 R_0) \dot{P}_l \tag{261}$$

$$B_r^2 = \frac{m\mu_2}{4\pi R_0 R r} \sum_{l=0}^{\infty} \frac{1}{\overline{\overline{D}}_{E,l}} (2l+1) l(l+1) h_l^{(1)}(k_2 r) j_l(k_1 R_0) P_l \tag{262}$$

$$B_\theta^2 = -\frac{m\mu_2 \sin\theta}{4\pi R_0 R r} \sum_{l=0}^{\infty} (2l+1) \frac{1}{\overline{\overline{D}}_{E,l}} [(l+1) h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] j_l(k_1 R_0) \dot{P}_l \tag{263}$$

$$B_\phi^2 = 0 \tag{264}$$

Tangential

(Interior)

$$E_r^1 = -\frac{\omega m\mu_1 k_1 \sin\theta \sin\phi}{4\pi r} \sum_{l=1}^{\infty} (2l+1) \frac{\overline{N}_{M,l}}{\overline{\overline{D}}_{M,l}} j_l(k_1 r) j_l(k_1 R_0) \dot{P}_l \tag{265}$$

$$E_\theta^1 = -\frac{\omega m\mu_1 k_1 \sin\phi}{4\pi R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \{ -\frac{\overline{N}_{E,l}}{\overline{\overline{D}}_{E,l}} r j_l(k_1 r) [(l+1) j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] \dot{P}_l \tag{266}$$

$$+ \frac{\overline{N}_{M,l}}{\overline{\overline{D}}_{M,l}} [(l+1) j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] R_0 j_l(k_1 R_0) [l(l+1) P_l - \cos\theta \dot{P}_l] \}$$

$$E_\phi^1 = \frac{\omega m\mu_1 k_1 \cos\phi}{4\pi R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \{ -\frac{\overline{N}_{M,l}}{\overline{\overline{D}}_{M,l}} R_0 j_l(k_1 R_0) [(l+1) j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] \dot{P}_l \tag{267}$$

$$+ \frac{\overline{N}_{E,l}}{\overline{\overline{D}}_{E,l}} r j_l(k_1 r) [(l+1) j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] [l(l+1) P_l - \cos\theta \dot{P}_l] \}$$

$$B_r^1 = \frac{i m\mu_1 k_1 \sin\theta \cos\phi}{4\pi R_0 r} \sum_{l=1}^{\infty} (2l+1) \frac{\overline{N}_{E,l}}{\overline{\overline{D}}_{E,l}} j_l(k_1 r) [(l+1) j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] \dot{P}_l \tag{268}$$

$$B_\theta^1 = \frac{i m\mu_1 k_1 \cos\phi}{4\pi R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \{ \frac{\overline{N}_{M,l}}{\overline{\overline{D}}_{M,l}} r R_0 k_1^2 j_l(k_1 r) j_l(k_1 R_0) \dot{P}_l \tag{269}$$

$$+ \frac{\overline{N}_{E,l}}{\overline{\overline{D}}_{E,l}} [(l+1) j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] [(l+1) j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] [l(l+1) P_l - \cos\theta \dot{P}_l] \}$$

$$B_\phi^1 = -\frac{i m\mu_1 k_1 \sin\phi}{4\pi R_0 r}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\{\frac{\overline{N}_{M,l}}{\overline{D}_{M,l}} r R_0 k_1^2 j_l(k_1 r) j_l(k_1 R_0)\,[l(l+1)P_l - \cos\theta \dot{P}_l] \tag{270}$$

$$+\frac{\overline{N}_{E,l}}{\overline{D}_{E,l}}[(l+1)j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)]\,[(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)]\dot{P}_l\}$$

(Exterior)

$$E_r^2 = -\frac{i\omega m\mu_2 \sin\theta\sin\phi}{4\pi R r}\sum_{l=1}^{\infty}(2l+1)\frac{1}{\overline{D}_{M,l}} h_l^{(1)}(k_2 r) j_l(k_1 R_0)\dot{P}_l \tag{271}$$

$$E_\theta^2 = -\frac{i\omega m\mu_2 \sin\phi}{4\pi R_0 R r}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\{\frac{1}{\overline{D}_{E,l}} r h_l^{(1)}(k_2 r)\,[(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)]\dot{P}_l \tag{272}$$

$$+\frac{1}{\overline{D}_{M,l}}[(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] R_0 j_l(k_1 R_0)\,[l(l+1)P_l - \cos\theta\dot{P}_l]\}$$

$$E_\phi^2 = -\frac{i\omega m\mu_2 \cos\phi}{4\pi R_0 R r}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\{\frac{1}{\overline{D}_{M,l}} R_0 j_l(k_1 R_0)\,[(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)]\dot{P}_l \tag{273}$$

$$+\frac{1}{\overline{D}_{E,l}} r h_l^{(1)}(k_2 r)\,[(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)]\,[l(l+1)P_l - \cos\theta\dot{P}_l]\}$$

$$B_r^2 = \frac{m\mu_2 \sin\theta\cos\phi}{4\pi R_0 R r}\sum_{l=1}^{\infty}(2l+1)\frac{1}{\overline{D}_{E,l}}[(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] h_l^{(1)}(k_2 r)\dot{P}_l \tag{274}$$

$$B_\theta^2 = \frac{m\mu_2 \cos\phi}{4\pi R_0 R r}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\{\frac{-1}{\overline{D}_{M,l}} r R_0 k_2^2 h_l^{(1)}(k_2 r) j_l(k_1 R_0)\dot{P}_l \tag{275}$$

$$+\frac{1}{\overline{D}_{E,l}}[(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)]\,[(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)]\,[l(l+1)P_l - \cos\theta\dot{P}_l]\}$$

$$B_\phi^2 = -\frac{m\mu_2 \sin\phi}{4\pi R_0 R r}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\{\frac{-1}{\overline{D}_{M,l}} r R_0 k_2^2 h_l^{(1)}(k_2 r) j_l(k_1 R_0)\,[l(l+1)P_l - \cos\theta\dot{P}_l] \tag{276}$$

$$+\frac{1}{\overline{D}_{E,l}}[(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)]\,[(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)]\dot{P}_l\}$$

CURRENT DIPOLE

Radial

(Interior)

$$E_r^1 = -\frac{\omega p \mu_1}{4\pi k_1 R_0 r} \sum_{l=0}^{\infty} \frac{\overline{N}_{M,l}}{\overline{D}_{M,l}} (2l+1) l (l+1) j_l(k_1 r) j_l(k_1 R_0) P_l \tag{277}$$

$$E_\theta^1 = \frac{\omega p \mu_1 \sin\theta}{4\pi k_1 R_0 r} \sum_{l=0}^{\infty} (2l+1) \frac{\overline{N}_{M,l}}{\overline{D}_{M,l}} [(l+1) j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] j_l(k_1 R_0) \dot{P}_l \tag{278}$$

$$E_\phi^1 = 0 \tag{279}$$

$$B_r^1 = 0 \tag{280}$$

$$B_\theta^1 = 0 \tag{281}$$

$$B_\phi^1 = \frac{i p \mu_1 k_1 \sin\theta}{4\pi R_0} \sum_{l=0}^{\infty} (2l+1) \frac{\overline{N}_{M,l}}{\overline{D}_{M,l}} j_l(k_1 r) j_l(k_1 R_0) \dot{P}_l \tag{282}$$

(Exterior)

$$E_r^2 = \frac{i \omega p \mu_2}{2\pi k_1^2 R_0 R r} \sum_{l=0}^{\infty} \frac{1}{\overline{D}_{M,l}} (2l+1) l (l+1) h_l^{(1)}(k_2 r) j_l(k_1 R_0) P_l \tag{283}$$

$$E_\theta^2 = -\frac{i \omega p \mu_2 \sin\theta}{2\pi k_1^2 R_0 R r} \sum_{l=0}^{\infty} (2l+1) \frac{1}{\overline{D}_{M,l}} [(l+1) h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] j_l(k_1 R_0) \dot{P}_l \tag{284}$$

$$E_\phi^2 = 0 \tag{285}$$

$$B_r^2 = 0 \tag{286}$$

$$B_\theta^2 = 0 \tag{287}$$

$$B_\phi^2 = \frac{p \mu_2 \bar{\gamma} \sin\theta}{2\pi R_0 R} \sum_{l=0}^{\infty} (2l+1) \frac{1}{\overline{D}_{M,l}} h_l^{(1)}(k_2 r) j_l(k_1 R_0) \dot{P}_l \tag{288}$$

Tangential

(Interior)

$$E_r^1 = -\frac{\omega p \mu_1 \sin\theta \cos\phi}{4\pi k_1 R_0 r} \sum_{l=1}^{\infty} (2l+1) \frac{\overline{N}_{M,l}}{\overline{D}_{M,l}} j_l(k_1 r) [(l+1) j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] \dot{P}_l \tag{289}$$

$$E_\theta^1 = -\frac{\omega p \mu_1 \cos\phi}{4\pi k_1 R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \{ \frac{\overline{N}_{E,l}}{\overline{D}_{E,l}} r R_0 k_1^2 j_l(k_1 r) j_l(k_1 R_0) \dot{P}_l$$

$$+ \frac{\overline{N}_{M,l}}{\overline{D}_{M,l}} [(l+1) j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] [(l+1) j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] [l(l+1) P_l - \cos\theta \dot{P}_l] \} \tag{290}$$

$$E_\phi^1 = \frac{\omega p \mu_1 \sin\phi}{4\pi k_1 R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \{ \frac{\overline{N}_{E,l}}{\overline{D}_{E,l}} r R_0 k_1^2 j_l(k_1 r) j_l(k_1 R_0) [l(l+1) P_l - \cos\theta \dot{P}_l]$$

$$+ \frac{\overline{N}_{M,l}}{\overline{D}_{M,l}} [(l+1) j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] [(l+1) j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] \dot{P}_l \} \tag{291}$$

$$B_r^1 = \frac{i p \mu_1 k_1 \sin\theta \sin\phi}{4\pi r} \sum_{l=1}^{\infty} (2l+1) \frac{\overline{N}_{E,l}}{\overline{D}_{E,l}} j_l(k_1 r) j_l(k_1 R_0) \dot{P}_l \tag{292}$$

$$B_\theta^1 = \frac{i p \mu_1 k_1 \sin\phi}{4\pi R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \{ -\frac{\overline{N}_{M,l}}{\overline{D}_{M,l}} r j_l(k_1 r) [(l+1) j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] \dot{P}_l \tag{293}$$

$$+ \frac{\overline{N}_{E,l}}{\overline{D}_{E,l}} R_0 j_l(k_1 R_0) [(l+1) j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] [l(l+1) P_l - \cos\theta \dot{P}_l] \}$$

$$B_\phi^1 = -\frac{i p \mu_1 k_1 \cos\phi}{4\pi R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \{ -\frac{\overline{N}_{E,l}}{\overline{D}_{E,l}} R_0 j_l(k_1 R_0) [(l+1) j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] \dot{P}_l$$

$$+ \frac{\overline{N}_{M,l}}{\overline{D}_{M,l}} r j_l(k_1 r) [(l+1) j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] [l(l+1) P_l - \cos\theta \dot{P}_l] \} \tag{294}$$

(Exterior)

$$E_r^2 = \frac{i \omega p \mu_2 \sin\theta \cos\phi}{2\pi k_1^2 R_0 R r} \sum_{l=1}^{\infty} (2l+1) \frac{1}{\overline{D}_{M,l}} h_l^{(1)}(k_2 r) [(l+1) j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] \dot{P}_l \tag{295}$$

$$E_\theta^2 = \frac{i\omega p\mu_2\cos\phi}{4\pi k_1^2 R_0 R r}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\{\frac{1}{\overline{D}_{E,l}}rR_0k_1^2h_l^{(1)}(k_2r)j_l(k_1R_0)\dot{P}_l \tag{296}$$

$$+\frac{2}{\overline{D}_{M,l}}[(l+1)h_l^{(1)}(k_2r)-k_2rh_{l+1}^{(1)}(k_2r)]\,[(l+1)j_l(k_1R_0)-k_1R_0j_{l+1}(k_1R_0)]\,[l(l+1)P_l-\cos\theta\dot{P}_l]\}$$

$$E_\phi^2 = -\frac{i\omega p\mu_2\sin\phi}{4\pi k_1^2 R_0 R r}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\{\frac{1}{\overline{D}_{E,l}}rR_0k_1^2h_l^{(1)}(k_2r)j_l(k_1R_0)\,[l(l+1)P_l-\cos\theta\dot{P}_l]$$

$$+\frac{2}{\overline{D}_{M,l}}[(l+1)h_l^{(1)}(k_2r)-k_2rh_{l+1}^{(1)}(k_2r)]\,[(l+1)j_l(k_1R_0)-k_1R_0j_{l+1}(k_1R_0)]\dot{P}_l\} \tag{297}$$

$$B_r^2 = \frac{p\mu_2\sin\theta\sin\phi}{4\pi R r}\sum_{l=1}^{\infty}(2l+1)\frac{1}{\overline{D}_{E,l}}h_l^{(1)}(k_2r)j_l(k_1R_0)\dot{P}_l \tag{298}$$

$$B_\theta^2 = \frac{p\mu_2\sin\phi}{4\pi R_0 R r}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\{-\frac{2\bar{\gamma}}{\overline{D}_{M,l}}rh_l^{(1)}(k_2r)\,[(l+1)j_l(k_1R_0)-k_1R_0j_{l+1}(k_1R_0)]\dot{P}_l$$

$$+\frac{1}{\overline{D}_{E,l}}[(l+1)h_l^{(1)}(k_2r)-k_2rh_{l+1}^{(1)}(k_2r)]R_0j_l(k_1R_0)\,[l(l+1)P_l-\cos\theta\dot{P}_l]\} \tag{299}$$

$$B_\phi^2 = \frac{p\mu_2\cos\phi}{4\pi R_0 R r}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\{\frac{1}{\overline{D}_{E,l}}R_0j_l(k_1R_0)\,[(l+1)h_l^{(1)}(k_2r)-k_2rh_{l+1}^{(1)}(k_2r)]\dot{P}_l$$

$$-\frac{2\bar{\gamma}}{\overline{D}_{M,l}}rh_l^{(1)}(k_2r)\,[(l+1)j_l(k_1R_0)-k_1R_0j_{l+1}(k_1R_0)]\,[l(l+1)P_l-\cos\theta\dot{P}_l]\} \tag{300}$$

DC LIMIT OF THE SCATTERED FIELDS FOR AN INTERNAL DIPOLE

The dc limits of the field expressions given in Equations (253) through (300) are not simply derived from the dc expressions for the external dipole, given in Equations (187) through (234). The limiting forms are derived directly and are given below. In writing the expressions, the results are given in terms of $\tau = \mu_1/\mu_2$ and $\delta = \sigma_1/\sigma_2$.

DC MAGNETIC DIPOLE

Radial

(Interior)

$$E_r^1 = 0 \tag{301}$$

$$E_\theta^1 = 0 \tag{302}$$

$$E_\phi^1 = 0 \tag{303}$$

$$B_r^1 = -\frac{m\mu_1(\tau-1)}{4\pi R_0 r R}\sum_{l=1}^{\infty}\frac{l^2(l+1)}{(\tau+1)l+1}\left(\frac{r}{R}\right)^l\left(\frac{R_0}{R}\right)^l P_l \tag{304}$$

$$B_\theta^1 = \frac{m\mu_1(\tau-1)\sin\theta}{4\pi R_0 r R}\sum_{l=1}^{\infty}\frac{l(l+1)}{(\tau+1)l+1}\left(\frac{r}{R}\right)^l\left(\frac{R_0}{R}\right)^l \dot{P}_l \tag{305}$$

$$B_\phi^1 = 0 \tag{306}$$

(Exterior)

$$E_r^2 = 0 \tag{307}$$

$$E_\theta^2 = 0 \tag{308}$$

$$E_\phi^2 = 0 \tag{309}$$

$$B_r^2 = \frac{m\mu_2\tau}{4\pi R_0 r^2}\sum_{l=1}^{\infty}\frac{(2l+1)l(l+1)}{(\tau+1)l+1}\left(\frac{R_0}{r}\right)^l P_l \tag{310}$$

$$B_\theta^2 = \frac{m\mu_2\tau\sin\theta}{4\pi R_0 r^2}\sum_{l=1}^{\infty}\frac{(2l+1)l}{(\tau+1)l+1}\left(\frac{R_0}{r}\right)^l \dot{P}_l \tag{311}$$

$$B_\phi^2 = 0 \tag{312}$$

Tangential

(Interior)

$$E_r^1 = 0 \tag{313}$$

$$E_\theta^1 = 0 \tag{314}$$

$$E_\phi^1 = 0 \tag{315}$$

$$B_r^1 = -\frac{m\mu_1(\tau-1)\sin\theta\cos\phi}{4\pi R_0 r R}\sum_{l=1}^{\infty}\frac{l(l+1)}{(\tau+1)l+1}\left(\frac{r}{R}\right)^l\left(\frac{R_0}{R}\right)^l \dot{P}_l \tag{316}$$

$$B_\theta^1 = -\frac{m\mu_1(\tau-1)\cos\phi}{4\pi R_0 rR}\sum_{l=1}^{\infty}\frac{(l+1)}{(\tau+1)l+1}\left(\frac{r}{R}\right)^l\left(\frac{R_0}{R}\right)^l[l(l+1)P_l-\cos\theta\dot{P}_l] \tag{317}$$

$$B_\phi^1 = \frac{m\mu_1(\tau-1)\sin\phi}{4\pi R_0 rR}\sum_{l=1}^{\infty}\frac{l+1}{(\tau+1)l+1}\left(\frac{r}{R}\right)^l\left(\frac{R_0}{R}\right)^l\dot{P}_l \tag{318}$$

(Exterior)

$$E_r^2 = 0 \tag{319}$$

$$E_\theta^2 = 0 \tag{320}$$

$$E_\phi^2 = 0 \tag{321}$$

$$B_r^2 = \frac{m\mu_2\tau\sin\theta\cos\phi}{4\pi R_0 r^2}\sum_{l=1}^{\infty}\frac{(2l+1)(l+1)}{(\tau+1)l+1}\left(\frac{R_0}{r}\right)^l\dot{P}_l \tag{322}$$

$$B_\theta^2 = -\frac{m\mu_2\tau\cos\phi}{4\pi R_0 r^2}\sum_{l=1}^{\infty}\frac{(2l+1)}{(\tau+1)l+1}\left(\frac{R_0}{r}\right)^l[l(l+1)P_l-\cos\theta\dot{P}_l] \tag{323}$$

$$B_\phi^2 = \frac{m\mu_2\tau\sin\phi}{4\pi R_0 r^2}\sum_{l=1}^{\infty}\frac{(2l+1)}{(\tau+1)l+1}\left(\frac{R_0}{r}\right)^l\dot{P}_l \tag{324}$$

DC CURRENT DIPOLE

Radial

(Interior)

$$E_r^1 = -\frac{p(\delta-1)}{4\pi\sigma_1 R_0 rR}\sum_{l=1}^{\infty}\frac{l^2(l+1)}{(\delta+1)l+1}\left(\frac{r}{R}\right)^l\left(\frac{R_0}{R}\right)^l P_l \tag{325}$$

$$E_\theta^1 = \frac{p(\delta-1)\sin\theta}{4\pi\sigma_1 R_0 rR}\sum_{l=1}^{\infty}\frac{l(l+1)}{(\delta+1)l+1}\left(\frac{r}{R}\right)^l\left(\frac{R_0}{R}\right)^l\dot{P}_l \tag{326}$$

$$E_\phi^1 = 0 \tag{327}$$

$$B_r^1 = 0 \tag{328}$$

$$B_\theta^1 = 0 \tag{329}$$

$$B_\phi^1 = -\frac{p(\delta-1)\mu_1 \sin\theta}{4\pi R_0 R}\sum_{l=1}^{\infty}\frac{l}{(\delta+1)l+1}\left(\frac{r}{R}\right)^l\left(\frac{R_0}{R}\right)^l \dot{P}_l \tag{330}$$

(Exterior)

$$E_r^2 = \frac{p}{2\pi\sigma_2 R_0 r^2}\sum_{l=1}^{\infty}\frac{(2l+1)l(l+1)}{(\delta+1)l+1}\left(\frac{R_0}{r}\right)^l P_l \tag{331}$$

$$E_\theta^2 = \frac{p\sin\theta}{2\pi\sigma_2 R_0 r^2}\sum_{l=1}^{\infty}\frac{(2l+1)l}{(\delta+1)l+1}\left(\frac{R_0}{r}\right)^l \dot{P}_l \tag{332}$$

$$E_\phi^2 = 0 \tag{333}$$

$$B_r^2 = 0 \tag{334}$$

$$B_\theta^2 = 0 \tag{335}$$

$$B_\phi^2 = \frac{p\mu_2 \sin\theta}{2\pi R_0 r}\sum_{l=1}^{\infty}\frac{(2l+1)}{(\delta+1)l+1}\left(\frac{R_0}{r}\right)^l \dot{P}_l \tag{336}$$

__Tangential__

(Interior)

$$E_r^1 = -\frac{p(\delta-1)\sin\theta\cos\phi}{4\pi\sigma_1 R_0 R r}\sum_{l=1}^{\infty}\frac{l(l+1)}{(\delta+1)l+1}\left(\frac{r}{R}\right)^l\left(\frac{R_0}{R}\right)^l \dot{P}_l \tag{337}$$

$$E_\theta^1 = -\frac{p(\delta-1)\cos\phi}{4\pi\sigma_1 R_0 R r}\sum_{l=1}^{\infty}\frac{l+1}{(\delta+1)l+1}\left(\frac{r}{R}\right)^l\left(\frac{R_0}{R}\right)^l [l(l+1)P_l - \cos\theta \dot{P}_l] \tag{338}$$

$$E_\phi^1 = \frac{p(\delta-1)\sin\phi}{4\pi\sigma_1 R_0 R r}\sum_{l=1}^{\infty}\frac{l+1}{(\delta+1)l+1}\left(\frac{r}{R}\right)^l\left(\frac{R_0}{R}\right)^l \dot{P}_l \tag{339}$$

$$B_r^1 = -\frac{p\mu_1(\tau-1)\sin\theta\sin\phi}{4\pi R r}\sum_{l=1}^{\infty}\frac{l}{(\tau+1)l+1}\left(\frac{r}{R}\right)^l\left(\frac{R_0}{R}\right)^l \dot{P}_l \tag{340}$$

$$B_\theta^1 = -\frac{p\mu_1 \sin\phi}{4\pi R R_0 r}\sum_{l=1}^{\infty}\left\{\frac{R_0(\tau-1)}{(\tau+1)l+1}[l(l+1)P_l - \cos\theta\dot{P}_l] - \frac{r(\delta-1)}{(\delta+1)l+1}\dot{P}_l\right\}\left(\frac{r}{R}\right)^l\left(\frac{R_0}{R}\right)^l \tag{341}$$

$$B_\phi^1 = -\frac{p\mu_1 \cos\phi}{4\pi R R_0 r}\sum_{l=1}^{\infty}\left\{\frac{R_0(\tau-1)}{(\tau+1)l+1}\dot{P}_l - \frac{r(\delta-1)}{(\delta+1)l+1}[l(l+1)P_l - \cos\theta \dot{P}_l]\right\}\left(\frac{r}{R}\right)^l\left(\frac{R_0}{R}\right)^l \quad (342)$$

(Exterior)

$$E_r^2 = \frac{p \sin\theta\cos\phi}{2\pi\sigma_2 R_0 r^2}\sum_{l=1}^{\infty}\frac{(2l+1)(l+1)}{(\delta+1)l+1}\left(\frac{R_0}{r}\right)^l \dot{P}_l \quad (343)$$

$$E_\theta^2 = -\frac{p\cos\phi}{2\pi\sigma_2 R_0 r^2}\sum_{l=1}^{\infty}\frac{(2l+1)}{(\delta+1)l+1}\left(\frac{R_0}{r}\right)^l [l(l+1)P_l - \cos\theta\dot{P}_l] \quad (344)$$

$$E_\phi^2 = \frac{p\sin\phi}{2\pi\sigma_2 R_0 r^2}\sum_{l=1}^{\infty}\frac{(2l+1)}{(\delta+1)l+1}\left(\frac{R_0}{r}\right)^l \dot{P}_l \quad (345)$$

$$B_r^2 = \frac{p\tau\mu_2 \sin\theta\sin\phi}{4\pi r^2}\sum_{l=1}^{\infty}\frac{2l+1}{(\tau+1)l+1}\left(\frac{R_0}{r}\right)^l \dot{P}_l \quad (346)$$

$$B_\theta^2 = \frac{p\mu_2\sin\phi}{4\pi R_0 r^2}\sum_{l=1}^{\infty}(2l+1)\left\{\frac{\tau R_0}{(l+1)[(\tau+1)l+1]}[l(l+1)P_l - \cos\theta\dot{P}_l] - \frac{2r}{l[(\delta+1)l+1]}\dot{P}_l\right\}\left(\frac{R_0}{r}\right)^l \quad (347)$$

$$B_\phi^2 = \frac{p\mu_2\cos\phi}{4\pi R_0 r^2}\sum_{l=1}^{\infty}(2l+1)\left\{\frac{\tau R_0}{(l+1)[(\tau+1)l+1]}\dot{P}_l - \frac{2r}{l[(\delta+1)l+1]}[l(l+1)P_l - \cos\theta\dot{P}_l]\right\}\left(\frac{R_0}{r}\right)^l \quad (348)$$

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APPENDIX A

NUMERICAL METHODS AND HEWLETT-PACKARD BASIC 3.0 COMPUTER CODES FOR CURRENT AND MAGNETIC DIPOLE FIELDS

APPENDIX A

NUMERICAL METHODS AND HEWLETT-PACKARD BASIC 3.0 COMPUTER CODES FOR CURRENT AND MAGNETIC DIPOLE FIELDS

In this Appendix, codes are given for computing the electric and magnetic fields external to a sphere, for dipole sources external to the sphere. Codes are given for ac sources(**MAGNDIPSPH** and **CURRDIPSPH**) and for dc sources (**MGDIPSPHDC** and **CRDIPSPHDC**). The Legendre polynomials and their first derivatives are generated in the subroutine **Pl_pldot** based on the recursion formulas.[A1] The spherical Bessel functions can be generated from closed-form expressions,[A1] but these have large roundoff errors for any given argument as the order increases. Instead, the Bessel functions are generated from various representations, depending on the values of argument and order. These routines were taken from Abramowitz and Stegun,[A1] and are discussed in detail below.

The structure of the remainder of this Appendix is as follows:

MAGNDIPSPH
CURRDIPSPH
MGDIPSPHDC
CRDIPSPHDC

The subroutines:

Geomdipsph
Geomfldpos
Geomfldb_e
Jcomb
Hcomb
Spherejnz
Spherehnz
Jnuevrywhr
Hnuevrywhr
Pl_pldot
Ndm
Nde
Ndmicon
Ndeicon

Discussion of series representations of Bessel functions.

The subroutines:

Jn
H1n
Jnu
H1nu
Prinlogz
Atn2
Gamma

Discussion of asymptotic representations of Bessel functions for large argument.

The subroutines:

Jnasy
H1nasy
Pnz
Qnz

Discussion of uniform asymptotic representation of Bessel functions for large order.

The subroutines:

Jfiuniasym
Hfkuniasym
Jh1produni
Uk
Ukcoefs

PROGRAM MAGNDIPSPH

```
10  OPTION BASE 1
20  DIM Xf(3),Xd(3),M(3),Rds(3,3),Bp(3,2),Ep(3,2),B(3,2),E(3,2)
30  DIM Pl(100),Pld(100),Ndma(100,2),Ndea(100,2),Et(3,2),Bt(3,2)
40  INPUT "FREQUENCY AND MEDIUM AND SPHERE(Ss<0 => INF)CONDUCTIVITIES(MHO/M)?",F,Sm,Ss
50  INPUT "RELATIVE PERMEABILITY OF MEDIUM AND SPHERE?",Mm,Ms
60  INPUT "RELATIVE PERMITTIVITY OF MEDIUM AND SPHERE?",Em,Es
70  INPUT "POSITION OF FIELD POINT?(M)",Xf(*)
80  INPUT "POSITION OF SOURCE POINT?(M)",Xd(*)
90  INPUT "MAGNETIC DIPOLE MOMENT VECTOR?(AMP-M^2)",M(*)
100 INPUT "SPHERE RADIUS(M) AND MAXIMUM POLAR ANGLE INDEX?",A,Ell
110 INPUT "RELATIVE ERROR FOR TRUNCATION?",Err
120 Lmax=Ell+1
130 W=2*PI*F
140 M0=4*PI*1.E-7
150 E0=8.85415E-12
160 Ut1=W*M0
170 Ut2=W*E0*Ut1
180 K12r=Ut2*Es*Ms
190 K12i=Ut1*Ms*Ss
200 K22r=Ut2*Em*Mm
210 K22i=Ut1*Mm*Sm
220 Md=SQR(K12r*K12r+K12i*K12i)
230 K1r=SQR((Md+K12r)/2)
240 K1i=SQR(ABS(Md-K12r)/2)
250 Md=SQR(K22r*K22r+K22i*K22i)
260 K2r=SQR((Md+K22r)/2)
270 K2i=SQR(ABS(Md-K22r)/2)
280 Mefac=W*M0*Mm/4/PI          ! Electric fields in volt/meter
290 Mefacr=Mefac*K2r
300 Mefaci=Mefac*K2i
310 Mmfac=M0*Mm/4/PI           ! Magnetic fields in tesla
```

```
320 Mmfacr=-Mmfac*K2i
330 Mmfaci=Mmfac*K2r
340 Mur=Ms/Mm
350  !Only frequency and medium constants to here
360 REDIM Ndma(Lmax,2),Ndea(Lmax,2)
370 IF Ss>0 THEN GOTO 410
380 CALL Ndmicon(Lmax,K2r,K2i,A,Ndma(*))
390 CALL Ndeicon(Lmax,K2r,K2i,A,Ndea(*))
400 GOTO 430
410 CALL Ndm(Lmax,K1r,K1i,K2r,K2i,Mur,A,Ndma(*))
420 CALL Nde(Lmax,K1r,K1i,K2r,K2i,Mur,A,Ndea(*))
430  !Added only sphere radius to here
440 CALL Geomdipsph(Xd(*),M(*),Rds(*),R0,Mr,Mt)
450 CALL Geomfldpos(Xf(*),Rds(*),R,Ct,St,Cp,Sp)
460 Zr=K2r*R
470 Zi=K2i*R
480 Z0r=K2r*R0
490 Z0i=K2i*R0
500 REDIM Pl(Lmax),Pld(Lmax)
510 CALL Pl_pldot(Lmax-1,Ct,Pl(*),Pld(*))
520 MAT Ep= (0)
530 MAT Bp= (0)
540 L=2
550 MAT Et= (0)
560 MAT Bt= (0)
570 CALL Hcomb(L-1,Zr,Zi,R,Hzr,Hzi,Chzr,Chzi)
580 CALL Hcomb(L-1,Z0r,Z0i,R0,Hz0r,Hz0i,Chz0r,Chz0i)
590 Te1r=Hzr*Hz0r-Hzi*Hz0i
600 Te1i=Hzr*Hz0i+Hzi*Hz0r
610 Te2r=Te1r*Ndma(L,1)-Te1i*Ndma(L,2)
620 Te2i=Te1r*Ndma(L,2)+Te1i*Ndma(L,1)
630 Tb2r=Te1r*Ndea(L,1)-Te1i*Ndea(L,2)
640 Tb2i=Te1r*Ndea(L,2)+Te1i*Ndea(L,1)
650 Et(1,1)=Mt*St*Sp*Te2r*Pld(L)/R
660 Et(1,2)=Mt*St*Sp*Te2i*Pld(L)/R
670 Te3r=Hz0r*Chzr-Hz0i*Chzi
680 Te3i=Hz0r*Chzi+Hz0i*Chzr
690 Te4r=Te3r*Ndma(L,1)-Te3i*Ndma(L,2)
700 Te4i=Te3r*Ndma(L,2)+Te3i*Ndma(L,1)
710 Te5r=Hzr*Chz0r-Hzi*Chz0i
720 Te5i=Hzr*Chz0i+Hzi*Chz0r
730 Te6r=Te5r*Ndea(L,1)-Te5i*Ndea(L,2)
740 Te6i=Te5r*Ndea(L,2)+Te5i*Ndea(L,1)
750 Bt(1,1)=Mr*Tb2r*L*(L-1)*Pl(L)/R/R0
760 Bt(1,2)=Mr*Tb2i*L*(L-1)*Pl(L)/R/R0
770 Bt(1,1)=Bt(1,1)+Mt*Cp*St*Te6r*Pld(L)/R
780 Bt(1,2)=Bt(1,2)+Mt*Cp*St*Te6i*Pld(L)/R
790 Tb3r=Chzr*Chz0r-Chzi*Chz0i
800 Tb3i=Chzr*Chz0i+Chzi*Chz0r
810 Tb4r=Tb3r*Ndea(L,1)-Tb3i*Ndea(L,2)
820 Tb4i=Tb3r*Ndea(L,2)+Tb3i*Ndea(L,1)
830 Tb5r=K22r*Te2r-K22i*Te2i
```

```
840  Tb5i=K22r*Te2i+K22i*Te2r
850  Tb6r=Te3r*Ndea(L,1)-Te3i*Ndea(L,2)
860  Tb6i=Te3r*Ndea(L,2)+Te3i*Ndea(L,1)
870  Et(2,1)=Et(2,1)+Mt*Sp*Te4r*(Pl(L)-Ct*Pld(L)/L/(L-1))
880  Et(2,2)=Et(2,2)+Mt*Sp*Te4i*(Pl(L)-Ct*Pld(L)/L/(L-1))
890  Et(2,1)=Et(2,1)-Mt*Sp*Te6r*Pld(L)/L/(L-1)
900  Et(2,2)=Et(2,2)-Mt*Sp*Te6i*Pld(L)/L/(L-1)
910  Bt(2,1)=Bt(2,1)-Mr*St*Tb6r*Pld(L)/R0
920  Bt(2,2)=Bt(2,2)-Mr*St*Tb6i*Pld(L)/R0
930  Bt(2,1)=Bt(2,1)+Mt*Cp*Tb5r*Pld(L)/L/(L-1)
940  Bt(2,2)=Bt(2,2)+Mt*Cp*Tb5i*Pld(L)/L/(L-1)
950  Bt(2,1)=Bt(2,1)+Mt*Cp*Tb4r*(Pl(L)-Ct*Pld(L)/L/(L-1))
960  Bt(2,2)=Bt(2,2)+Mt*Cp*Tb4i*(Pl(L)-Ct*Pld(L)/L/(L-1))
970  Te7r=Te1r*Ndea(L,1)-Te1i*Ndea(L,2)
980  Te7i=Te1r*Ndea(L,2)+Te1i*Ndea(L,1)
990  Te8r=Te3r*Ndma(L,1)-Te3i*Ndma(L,2)
1000 Te8i=Te3r*Ndma(L,2)+Te3i*Ndma(L,1)
1010 Et(3,1)=Et(3,1)+Mt*Cp*Te6r*(Pl(L)-Ct*Pld(L)/L/(L-1))
1020 Et(3,2)=Et(3,2)+Mt*Cp*Te6i*(Pl(L)-Ct*Pld(L)/L/(L-1))
1030 Et(3,1)=Et(3,1)-Mt*Cp*Te8r*Pld(L)/L/(L-1)
1040 Et(3,2)=Et(3,2)-Mt*Cp*Te8i*Pld(L)/L/(L-1)
1050 Et(3,1)=Et(3,1)-Mr*Te7r*St*Pld(L)/R0
1060 Et(3,2)=Et(3,2)-Mr*Te7i*St*Pld(L)/R0
1070 Bt(3,1)=Bt(3,1)+Mt*Sp*Tb5r*(Pl(L)-Ct*Pld(L)/L/(L-1))
1080 Bt(3,2)=Bt(3,2)+Mt*Sp*Tb5i*(Pl(L)-Ct*Pld(L)/L/(L-1))
1090 Bt(3,1)=Bt(3,1)+Mt*Sp*Tb4r*Pld(L)/L/(L-1)
1100 Bt(3,2)=Bt(3,2)+Mt*Sp*Tb4i*Pld(L)/L/(L-1)
1110 Ep(1,1)=Ep(1,1)+(2*L-1)*Et(1,1)
1120 Ep(1,2)=Ep(1,2)+(2*L-1)*Et(1,2)
1130 Ep(2,1)=Ep(2,1)+(2*L-1)*Et(2,1)
1140 Ep(2,2)=Ep(2,2)+(2*L-1)*Et(2,2)
1150 Ep(3,1)=Ep(3,1)+(2*L-1)*Et(3,1)
1160 Ep(3,2)=Ep(3,2)+(2*L-1)*Et(3,2)
1170 Bp(1,1)=Bp(1,1)+(2*L-1)*Bt(1,1)
1180 Bp(1,2)=Bp(1,2)+(2*L-1)*Bt(1,2)
1190 Bp(2,1)=Bp(2,1)+(2*L-1)*Bt(2,1)
1200 Bp(2,2)=Bp(2,2)+(2*L-1)*Bt(2,2)
1210 Bp(3,1)=Bp(3,1)+(2*L-1)*Bt(3,1)
1220 Bp(3,2)=Bp(3,2)+(2*L-1)*Bt(3,2)
1230 Ner1=Et(1,1)*Et(1,1)+Et(1,2)*Et(1,2)
1240 Ner2=Et(2,1)*Et(2,1)+Et(2,2)*Et(2,2)
1250 Ner3=Et(3,1)*Et(3,1)+Et(3,2)*Et(3,2)
1260 Dne1=Ep(1,1)*Ep(1,1)+Ep(1,2)*Ep(1,2)
1270 Dne2=Ep(2,1)*Ep(2,1)+Ep(2,2)*Ep(2,2)
1280 Dne3=Ep(3,1)*Ep(3,1)+Ep(3,2)*Ep(3,2)
1290 Nbr1=Bt(1,1)*Bt(1,1)+Bt(1,2)*Bt(1,2)
1300 Nbr2=Bt(2,1)*Bt(2,1)+Bt(2,2)*Bt(2,2)
1310 Nbr3=Bt(3,1)*Bt(3,1)+Bt(3,2)*Bt(3,2)
1320 Dnb1=Bp(1,1)*Bp(1,1)+Bp(1,2)*Bp(1,2)
1330 Dnb2=Bp(2,1)*Bp(2,1)+Bp(2,2)*Bp(2,2)
1340 Dnb3=Bp(3,1)*Bp(3,1)+Bp(3,2)*Bp(3,2)
1350 IF L=Lmax THEN 1570
```

```
1360 IF Dnb1=0 THEN 1380
1370 IF Nbr1/Dnb1<Err*Err THEN
1380  IF Dnb2=0 THEN 1400
1390  IF Nbr2/Dnb2<Err*Err THEN
1400   IF Dnb3=0 THEN 1420
1410   IF Nbr3/Dnb3<Err*Err THEN
1420    IF Dne1=0 THEN 1440
1430    IF Ner1/Dne1<Err*Err THEN
1440     IF Dne2=0 THEN 1460
1450     IF Ner2/Dne2<Err*Err THEN
1460      IF Dne3=0 THEN 1480
1470      IF Ner3/Dne3<Err*Err THEN
1480       GOTO 1570
1490      END IF
1500     END IF
1510    END IF
1520   END IF
1530  END IF
1540 END IF
1550 L=L+1
1560 GOTO 550
1570 Tmpr=Mefacr*Ep(1,1)-Mefaci*Ep(1,2)
1580 Tmpi=Mefacr*Ep(1,2)+Mefaci*Ep(1,1)
1590 Ep(1,1)=-Tmpr
1600 Ep(1,2)=-Tmpi
1610 Tmpr=Mmfacr*Bp(1,1)-Mmfaci*Bp(1,2)
1620 Tmpi=Mmfacr*Bp(1,2)+Mmfaci*Bp(1,1)
1630 Bp(1,1)=Tmpr
1640 Bp(1,2)=Tmpi
1650 Tmpr=Mefacr*Ep(2,1)-Mefaci*Ep(2,2)
1660 Tmpi=Mefacr*Ep(2,2)+Mefaci*Ep(2,1)
1670 Ep(2,1)=-Tmpr
1680 Ep(2,2)=-Tmpi
1690 Tmpr=Mmfacr*Bp(2,1)-Mmfaci*Bp(2,2)
1700 Tmpi=Mmfacr*Bp(2,2)+Mmfaci*Bp(2,1)
1710 Bp(2,1)=Tmpr
1720 Bp(2,2)=Tmpi
1730 Tmpr=Mefacr*Ep(3,1)-Mefaci*Ep(3,2)
1740 Tmpi=Mefacr*Ep(3,2)+Mefaci*Ep(3,1)
1750 Ep(3,1)=Tmpr
1760 Ep(3,2)=Tmpi
1770 Tmpr=Mmfacr*Bp(3,1)-Mmfaci*Bp(3,2)
1780 Tmpi=Mmfacr*Bp(3,2)+Mmfaci*Bp(3,1)
1790 Bp(3,1)=-Tmpr
1800 Bp(3,2)=-Tmpi
1810 CALL Geomfldb_e(Ct,St,Cp,Sp,Bp(*),Ep(*),Rds(*),B(*),E(*))
1820 PRINT E(*)  !Electric field in original frame
1830 PRINT
1840 PRINT B(*)  !Magnetic field in original frame
1850 END
```

PROGRAM CURRDIPSPH

```
10  OPTION BASE 1
20  DIM Xf(3),Xd(3),P(3),Rds(3,3),Bp(3,2),Ep(3,2),B(3,2),E(3,2)
30  DIM Pl(100),Pld(100),Ndma(100,2),Ndea(100,2),Et(3,2),Bt(3,2)
40      INPUT "FREQUENCY AND MEDIUM AND SPHERE(Ss<0 => INF)
CONDUCTIVITIES(MHO/M)?",F,Sm,Ss
50  INPUT "RELATIVE PERMEABILITY OF MEDIUM AND SPHERE?",Mm,Ms
60  INPUT "RELATIVE PERMITTIVITY OF MEDIUM AND SPHERE?",Em,Es
70  INPUT "POSITION OF FIELD POINT?(M)",Xf(*)
80  INPUT "POSITION OF SOURCE POINT?(M)",Xd(*)
90  INPUT "CURRENT DIPOLE MOMENT VECTOR?(AMP-M)",P(*)
100 INPUT " SPHERE RADIUS(M) AND MAXIMUM POLAR ANGLE INDEX?",A,Ell
110 Err=1.E-6
120 Lmax=Ell+1
130 W=2*PI*F
140 M0=4*PI*1.E-7
150 E0=8.85415E-12
160 Ut1=W*M0
170 Ut2=W*E0*Ut1
180 K12r=Ut2*Es*Ms
190 K12i=Ut1*Ms*Ss
200 K22r=Ut2*Em*Mm
210 K22i=Ut1*Mm*Sm
220 Md=SQR(K12r*K12r+K12i*K12i)
230 K1r=SQR((Md+K12r)/2)
240 K1i=SQR(ABS(Md-K12r)/2)
250 Md=SQR(K22r*K22r+K22i*K22i)
260 K2r=SQR((Md+K22r)/2)
270 K2i=SQR(ABS(Md-K22r)/2)
280 Mk2=K2r*K2r+K2i*K2i
290 Ik2r=K2r/Mk2
300 Ik2i=-K2i/Mk2
310 Eefac=M0*Mm*W/4/PI          !Electric fields in volt/meter
320 Eefacr=Eefac*Ik2r
330 Eefaci=Eefac*Ik2i
340 Emfac=M0*Mm/4/PI            !Magnetic fields in tesla
350 Emfacr=-Emfac*K2i
360 Emfaci=Emfac*K2r
370 Mur=Ms/Mm
380  !Only frequency and medium constants to here
390 REDIM Ndma(Lmax,2),Ndea(Lmax,2)
400 IF Ss>0 THEN 440
410 CALL Ndmicon(Lmax,K2r,K2i,A,Ndma(*))
420 CALL Ndeicon(Lmax,K2r,K2i,A,Ndea(*))
430 GOTO 460
440 CALL Ndm(Lmax,K1r,K1i,K2r,K2i,Mur,A,Ndma(*))
450 CALL Nde(Lmax,K1r,K1i,K2r,K2i,Mur,A,Ndea(*))
460  !Added only sphere radius to here
470 CALL Geomdipsph(Xd(*),P(*),Rds(*),R0,Pr,Pt)
480 CALL Geomfldpos(Xf(*),Rds(*),R,Ct,St,Cp,Sp)
490 Zr=K2r*R
```

```
500  Zi=K2i*R
510  Z0r=K2r*R0
520  Z0i=K2i*R0
530  REDIM Pl(Lmax),Pld(Lmax)
540  CALL Pl_pldot(Lmax-1,Ct,Pl(*),Pld(*))
550  MAT Ep= (0)
560  MAT Bp= (0)
570  L=2
580  MAT Et= (0)
590  MAT Bt= (0)
600  CALL Hcomb(L-1,Zr,Zi,R,Hzr,Hzi,Chzr,Chzi)
610  CALL Hcomb(L-1,Z0r,Z0i,R0,Hz0r,Hz0i,Chz0r,Chz0i)
620  Tb1r=Hzr*Hz0r-Hzi*Hz0i
630  Tb1i=Hzr*Hz0i+Hzi*Hz0r
640  Tb2r=Tb1r*Ndea(L,1)-Tb1i*Ndea(L,2)
650  Tb2i=Tb1r*Ndea(L,2)+Tb1i*Ndea(L,1)
660  Te2r=Tb1r*Ndma(L,1)-Tb1i*Ndma(L,2)
670  Te2i=Tb1r*Ndma(L,2)+Tb1i*Ndma(L,1)
680  Bt(1,1)=Pt*St*Sp*Tb2r*Pld(L)/R
690  Bt(1,2)=Pt*St*Sp*Tb2i*Pld(L)/R
700  Tb3r=Hz0r*Chzr-Hz0i*Chzi
710  Tb3i=Hz0r*Chzi+Hz0i*Chzr
720  Tb4r=Tb3r*Ndea(L,1)-Tb3i*Ndea(L,2)
730  Tb4i=Tb3r*Ndea(L,2)+Tb3i*Ndea(L,1)
740  Tb5r=Hzr*Chz0r-Hzi*Chz0i
750  Tb5i=Hzr*Chz0i+Hzi*Chz0r
760  Tb6r=Tb5r*Ndma(L,1)-Tb5i*Ndma(L,2)
770  Tb6i=Tb5r*Ndma(L,2)+Tb5i*Ndma(L,1)
780  Et(1,1)=Pr*Te2r*L*(L-1)*Pl(L)/R/R0
790  Et(1,2)=Pr*Te2i*L*(L-1)*Pl(L)/R/R0
800  Et(1,1)=Et(1,1)+Pt*Cp*St*Tb6r*Pld(L)/R
810  Et(1,2)=Et(1,2)+Pt*Cp*St*Tb6i*Pld(L)/R
820  Te3r=Chzr*Chz0r-Chzi*Chz0i
830  Te3i=Chzr*Chz0i+Chzi*Chz0r
840  Te4r=Te3r*Ndma(L,1)-Te3i*Ndma(L,2)
850  Te4i=Te3r*Ndma(L,2)+Te3i*Ndma(L,1)
860  Te5r=-K22r*Tb2r+K22i*Tb2i
870  Te5i=-K22r*Tb2i-K22i*Tb2r
880  Te6r=Tb3r*Ndma(L,1)-Tb3i*Ndma(L,2)
890  Te6i=Tb3r*Ndma(L,2)+Tb3i*Ndma(L,1)
900  Bt(2,1)=Bt(2,1)+Pt*Sp*Tb4r*(Pl(L)-Ct*Pld(L)/L/(L-1))
910  Bt(2,2)=Bt(2,2)+Pt*Sp*Tb4i*(Pl(L)-Ct*Pld(L)/L/(L-1))
920  Bt(2,1)=Bt(2,1)-Pt*Sp*Tb6r*Pld(L)/L/(L-1)
930  Bt(2,2)=Bt(2,2)-Pt*Sp*Tb6i*Pld(L)/L/(L-1)
940  Et(2,1)=Et(2,1)+Pr*St*Te6r*Pld(L)/R0
950  Et(2,2)=Et(2,2)+Pr*St*Te6i*Pld(L)/R0
960  Et(2,1)=Et(2,1)+Pt*Cp*Te5r*Pld(L)/L/(L-1)
970  Et(2,2)=Et(2,2)+Pt*Cp*Te5i*Pld(L)/L/(L-1)
980  Et(2,1)=Et(2,1)-Pt*Cp*Te4r*(Pl(L)-Ct*Pld(L)/L/(L-1))
990  Et(2,2)=Et(2,2)-Pt*Cp*Te4i*(Pl(L)-Ct*Pld(L)/L/(L-1))
1000 Tb7r=Tb1r*Ndma(L,1)-Tb1i*Ndma(L,2)
1010 Tb7i=Tb1r*Ndma(L,2)+Tb1i*Ndma(L,1)
```

```
1020 Tb8r=Tb3r*Ndea(L,1)-Tb3i*Ndea(L,2)
1030 Tb8i=Tb3r*Ndea(L,2)+Tb3i*Ndea(L,1)
1040 Bt(3,1)=Bt(3,1)-Pt*Cp*Tb6r*(Pl(L)-Ct*Pld(L)/L/(L-1))
1050 Bt(3,2)=Bt(3,2)-Pt*Cp*Tb6i*(Pl(L)-Ct*Pld(L)/L/(L-1))
1060 Bt(3,1)=Bt(3,1)+Pt*Cp*Tb8r*Pld(L)/L/(L-1)
1070 Bt(3,2)=Bt(3,2)+Pt*Cp*Tb8i*Pld(L)/L/(L-1)
1080 Bt(3,1)=Bt(3,1)+Pr*Tb7r*St*Pld(L)/R0
1090 Bt(3,2)=Bt(3,2)+Pr*Tb7i*St*Pld(L)/R0
1100 Et(3,1)=Et(3,1)-Pt*Sp*Te5r*(Pl(L)-Ct*Pld(L)/L/(L-1))
1110 Et(3,2)=Et(3,2)-Pt*Sp*Te5i*(Pl(L)-Ct*Pld(L)/L/(L-1))
1120 Et(3,1)=Et(3,1)+Pt*Sp*Te4r*Pld(L)/L/(L-1)
1130 Et(3,2)=Et(3,2)+Pt*Sp*Te4i*Pld(L)/L/(L-1)
1140 Bp(1,1)=Bp(1,1)+(2*L-1)*Bt(1,1)
1150 Bp(1,2)=Bp(1,2)+(2*L-1)*Bt(1,2)
1160 Bp(2,1)=Bp(2,1)+(2*L-1)*Bt(2,1)
1170 Bp(2,2)=Bp(2,2)+(2*L-1)*Bt(2,2)
1180 Bp(3,1)=Bp(3,1)+(2*L-1)*Bt(3,1)
1190 Bp(3,2)=Bp(3,2)+(2*L-1)*Bt(3,2)
1200 Ep(1,1)=Ep(1,1)-(2*L-1)*Et(1,1)
1210 Ep(1,2)=Ep(1,2)-(2*L-1)*Et(1,2)
1220 Ep(2,1)=Ep(2,1)+(2*L-1)*Et(2,1)
1230 Ep(2,2)=Ep(2,2)+(2*L-1)*Et(2,2)
1240 Ep(3,1)=Ep(3,1)+(2*L-1)*Et(3,1)
1250 Ep(3,2)=Ep(3,2)+(2*L-1)*Et(3,2)
1260 Ner1=Et(1,1)*Et(1,1)+Et(1,2)*Et(1,2)
1270 Ner2=Et(2,1)*Et(2,1)+Et(2,2)*Et(2,2)
1280 Ner3=Et(3,1)*Et(3,1)+Et(3,2)*Et(3,2)
1290 Dne1=Ep(1,1)*Ep(1,1)+Ep(1,2)*Ep(1,2)
1300 Dne2=Ep(2,1)*Ep(2,1)+Ep(2,2)*Ep(2,2)
1310 Dne3=Ep(3,1)*Ep(3,1)+Ep(3,2)*Ep(3,2)
1320 Nbr1=Bt(1,1)*Bt(1,1)+Bt(1,2)*Bt(1,2)
1330 Nbr2=Bt(2,1)*Bt(2,1)+Bt(2,2)*Bt(2,2)
1340 Nbr3=Bt(3,1)*Bt(3,1)+Bt(3,2)*Bt(3,2)
1350 Dnb1=Bp(1,1)*Bp(1,1)+Bp(1,2)*Bp(1,2)
1360 Dnb2=Bp(2,1)*Bp(2,1)+Bp(2,2)*Bp(2,2)
1370 Dnb3=Bp(3,1)*Bp(3,1)+Bp(3,2)*Bp(3,2)
1380 IF L=Lmax THEN 1600
1390 IF Dnb1=0 THEN 1410
1400 IF Nbr1/Dnb1<Err*Err THEN
1410   IF Dnb2=0 THEN 1430
1420   IF Nbr2/Dnb2<Err*Err THEN
1430     IF Dnb3=0 THEN 1450
1440     IF Nbr3/Dnb3<Err*Err THEN
1450       IF Dne1=0 THEN 1470
1460       IF Ner1/Dne1<Err*Err THEN
1470         IF Dne2=0 THEN 1490
1480         IF Ner2/Dne2<Err*Err THEN
1490           IF Dne3=0 THEN 1510
1500           IF Ner3/Dne3<Err*Err THEN
1510             GOTO 1600
1520           END IF
1530         END IF
```

```
1540      END IF
1550     END IF
1560    END IF
1570 END IF
1580 L=L+1
1590 GOTO 580
1600 Tmpr=Emfacr*Bp(1,1)-Emfaci*Bp(1,2)
1610 Tmpi=Emfacr*Bp(1,2)+Emfaci*Bp(1,1)
1620 Bp(1,1)=Tmpr
1630 Bp(1,2)=Tmpi
1640 Tmpr=Eefacr*Ep(1,1)-Eefaci*Ep(1,2)
1650 Tmpi=Eefacr*Ep(1,2)+Eefaci*Ep(1,1)
1660 Ep(1,1)=Tmpr
1670 Ep(1,2)=Tmpi
1680 Tmpr=Emfacr*Bp(2,1)-Emfaci*Bp(2,2)
1690 Tmpi=Emfacr*Bp(2,2)+Emfaci*Bp(2,1)
1700 Bp(2,1)=Tmpr
1710 Bp(2,2)=Tmpi
1720 Tmpr=Eefacr*Ep(2,1)-Eefaci*Ep(2,2)
1730 Tmpi=Eefacr*Ep(2,2)+Eefaci*Ep(2,1)
1740 Ep(2,1)=Tmpr
1750 Ep(2,2)=Tmpi
1760 Tmpr=Emfacr*Bp(3,1)-Emfaci*Bp(3,2)
1770 Tmpi=Emfacr*Bp(3,2)+Emfaci*Bp(3,1)
1780 Bp(3,1)=Tmpr
1790 Bp(3,2)=Tmpi
1800 Tmpr=Eefacr*Ep(3,1)-Eefaci*Ep(3,2)
1810 Tmpi=Eefacr*Ep(3,2)+Eefaci*Ep(3,1)
1820 Ep(3,1)=Tmpr
1830 Ep(3,2)=Tmpi
1840 CALL Geomfldb_e(Ct,St,Cp,Sp,Bp(*),Ep(*),Rds(*),B(*),E(*))
1850 PRINT E(*)  !Electric field in original frame
1860 PRINT
1870 PRINT B(*)  !Magnetic field in original frame
1880 END
```

PROGRAM MGDIPSPHDC

```
10   OPTION BASE 1
20   DIM Xf(3),Xd(3),M(3),Rds(3,3),Bp(3),Ep(3),B(3),E(3)
30   DIM Pl(100),Pld(100),Et(3),Bt(3)
40   INPUT "MEDIUM AND SPHERE CONDUCTIVITIES(MHO/M)?",Sm,Ss
50   INPUT "RELATIVE PERMEABILITY OF MEDIUM AND SPHERE?",Mm,Ms
60   INPUT "POSITION OF FIELD POINT?(M)",Xf(*)
70   INPUT "POSITION OF SOURCE POINT?(M)",Xd(*)
80   INPUT "MAGNETIC DIPOLE MOMENT VECTOR?(AMP-M^2)",M(*)
90   INPUT " SPHERE RADIUS(M) AND MAXIMUM POLAR ANGLE INDEX?",A,Ell
100  Err=1.E-6
110  Lmax=Ell+1
120  T=Ms/Mm
130  M0=4*PI*1.E-7
```

```
140  !Only medium constants to here
150  CALL Geomdipsph(Xd(*),M(*),Rds(*),R0,Mr,Mt)
160  CALL Geomfldpos(Xf(*),Rds(*),R,Ct,St,Cp,Sp)
170  Facm=M0*Mm*(T-1)/4/PI/A/R/R0
180  REDIM Pl(Lmax),Pld(Lmax)
190  CALL Pl_pldot(Lmax-1,Ct,Pl(*),Pld(*))
200  MAT Ep= (0)
210  MAT Bp= (0)
220  Rat=A*A/R/R0
230  Ratl=Rat
240  L=2
250  MAT Bt= (0)
260  Ratl=Ratl*Rat
270  Dt=1/((T+1)*(L-1)+1)
280  Bt(1)=-Mt*St*Cp*L*(L-1)*Ratl*Pld(L)*Dt
290  Bt(1)=Bt(1)+Mr*L*L*(L-1)*Ratl*Pl(L)*Dt
300  Bt(2)=Mt*Cp*(L-1)*Ratl*(L*(L-1)*Pl(L)-Ct*Pld(L))*Dt
310  Bt(2)=Bt(2)+Mr*St*L*(L-1)*Pld(L)*Ratl*Dt
320  Bt(3)=-Mt*Sp*(L-1)*Ratl*Pld(L)*Dt
330  MAT Bp= Bp+Bt
340  Nbr1=Bt(1)*Bt(1)
350  Nbr2=Bt(2)*Bt(2)
360  Nbr3=Bt(3)*Bt(3)
370  Dnb1=Bp(1)*Bp(1)
380  Dnb2=Bp(2)*Bp(2)
390  Dnb3=Bp(3)*Bp(3)
400  IF L=Lmax THEN 530
410  IF Dnb1=0 THEN 430
420  IF Nbr1/Dnb1<Err*Err THEN
430    IF Dnb2=0 THEN 450
440    IF Nbr2/Dnb2<Err*Err THEN
450      IF Dnb3=0 THEN 470
460      IF Nbr3/Dnb3<Err*Err THEN
470        GOTO 530
480      END IF
490    END IF
500  END IF
510  L=L+1
520  GOTO 250
530  Bp(1)=Facm*Bp(1)
540  Bp(2)=Facm*Bp(2)
550  Bp(3)=Facm*Bp(3)
560  CALL Geomfldb_e(Ct,St,Cp,Sp,Bp(*),Ep(*),Rds(*),B(*),E(*))
570  PRINT E(*)  !DC electric field in original frame(volt/meter)
580  PRINT
590  PRINT B(*)  !DC magnetic field in original frame(tesla)
600  END
```

PROGRAM CRDIPSPHDC

```
10  OPTION BASE 1
20  DIM Xf(3),Xd(3),P(3),Rds(3,3),Bp(3),Ep(3),B(3),E(3)
30  DIM Pl(100),Pld(100),Et(3),Bt(3)
40  INPUT "MEDIUM AND SPHERE CONDUCTIVITIES(MHO/M)?",Sm,Ss
50  INPUT "RELATIVE PERMEABILITY OF MEDIUM AND SPHERE?",Mm,Ms
60  INPUT "POSITION OF FIELD POINT?(M)",Xf(*)
70  INPUT "POSITION OF SOURCE POINT?(M)",Xd(*)
80  INPUT "CURRENT DIPOLE MOMENT VECTOR?(AMP-M)",P(*)
90  INPUT " SPHERE RADIUS(M) AND MAXIMUM POLAR ANGLE INDEX?",A,Ell
100 Err=1.E-6
110 Lmax=Ell+1
120 M0=4*PI*1.E-7
130 Facm=M0*Mm/4/PI/A
140 Face=1/4/PI/A/Sm
141 T=Ms/Mm
150 Ee=Ss/Sm
170  !Only medium constants to here
180 CALL Geomdipsph(Xd(*),P(*),Rds(*),R0,Pr,Pt)
190 CALL Geomfldpos(Xf(*),Rds(*),R,Ct,St,Cp,Sp)
200 REDIM Pl(Lmax),Pld(Lmax)
210 CALL Pl_pldot(Lmax-1,Ct,Pl(*),Pld(*))
220 MAT Ep= (0)
230 MAT Bp= (0)
240 Rat=A*A/R/R0
250 Ratl=Rat
260 L=2
270 MAT Et= (0)
280 MAT Bt= (0)
290 Ratl=Ratl*Rat
300 Dt=1/((T+1)*(L-1)+1)
310 De=1/((Ee+1)*(L-1)+1)
320 Bt(1)=(T-1)*Pt*St*Sp*L*Ratl*Pld(L)*Dt
330 Et(1)=(Ee-1)*Pr*L*L*(L-1)*Pl(L)*Ratl*De
340 Et(1)=Et(1)-(Ee-1)*Pt*Cp*St*L*(L-1)*Ratl*Pld(L)*De
350 Bt(2)=Pt*Sp*(Ee-1)*Ratl*Pld(L)/R0*De
360 Bt(2)=Bt(2)-Pt*Sp*(T-1)*(L*(L-1)*Pl(L)-Ct*Pld(L))*Ratl/R*Dt
370 Et(2)=(Ee-1)*Pr*St*L*(L-1)*Ratl*Pld(L)*De
380 Et(2)=Et(2)+(Ee-1)*Pt*Cp*(L-1)*Ratl*(L*(L-1)*Pl(L)-Ct*Pld(L))*De
390 Bt(3)=(Ee-1)*Pr*St*L*Ratl*Pld(L)*De/R0
400 Bt(3)=Bt(3)+(Ee-1)*Pt*Cp*(L*(L-1)*Pl(L)-Ct*Pld(L))*Ratl/R0*De
410 Bt(3)=Bt(3)-(T-1)*Pt*Cp*Ratl*Pld(L)/R*Dt
420 Et(3)=-Pt*Sp*(Ee-1)*(L-1)*Ratl*Pld(L)*De
430 MAT Bp= Bp+Bt
440 MAT Ep= Ep+Et
450 Ner1=Et(1)*Et(1)
460 Ner2=Et(2)*Et(2)
470 Ner3=Et(3)*Et(3)
480 Dne1=Ep(1)*Ep(1)
490 Dne2=Ep(2)*Ep(2)
500 Dne3=Ep(3)*Ep(3)
```

```
510  Nbr1=Bt(1)*Bt(1)
520  Nbr2=Bt(2)*Bt(2)
530  Nbr3=Bt(3)*Bt(3)
540  Dnb1=Bp(1)*Bp(1)
550  Dnb2=Bp(2)*Bp(2)
560  Dnb3=Bp(3)*Bp(3)
570  IF L=Lmax THEN 790
580  IF Dnb1=0 THEN 600
590  IF Nbr1/Dnb1<Err*Err THEN
600   IF Dnb2=0 THEN 620
610   IF Nbr2/Dnb2<Err*Err THEN
620    IF Dnb3=0 THEN 640
630    IF Nbr3/Dnb3<Err*Err THEN
640     IF Dne1=0 THEN 660
650     IF Ner1/Dne1<Err*Err THEN
660      IF Dne2=0 THEN 680
670      IF Ner2/Dne2<Err*Err THEN
680       IF Dne3=0 THEN 700
690       IF Ner3/Dne3<Err*Err THEN
700        GOTO 790
710       END IF
720      END IF
730     END IF
740    END IF
750   END IF
760  END IF
770  L=L+1
780  GOTO 270
790  Bp(1)=Facm*Bp(1)/R
800  Ep(1)=Face*Ep(1)/R/R0
810  Bp(2)=Facm*Bp(2)
820  Ep(2)=Face*Ep(2)/R/R0
830  Bp(3)=Facm*Bp(3)
840  Ep(3)=Face*Ep(3)/R0/R
850  CALL Geomfldb_e(Ct,St,Cp,Sp,Bp(*),Ep(*),Rds(*),B(*),E(*))
860  PRINT E(*)  !DC electric field in original frame(volt/meter)
870  PRINT
880  PRINT B(*)  !DC magnetic field in original frame(tesla)
890  END

10   SUB Geomdipsph(D(*),V(*),R(*),Dm,Vr,Vt)
20    !
30    !Produces transformation R(*) from general frame with source com-
40    !ponents D(*) and moment components V(*) to the standard frame
50    !with source on the z-axis, radial component Vr, and tangential
60    !component Vt oriented along the positive x-axis.
70    !
80    OPTION BASE 1
90    DIM R1(3,3),R2(3,3),R3(3,3),Rt(3,3),Vtm(3)
100    MAT R1= (0)
110    MAT R2= (0)
```

```
120   MAT R3= (0)
130   Dh=SQR(D(1)*D(1)+D(2)*D(2))
140   Dm=SQR(Dh*Dh+D(3)*D(3))
150   Std=Dh/Dm
160   Ctd=D(3)/Dm
170   IF Std>1.E-12 THEN
180     Cpd=D(1)/Dh
190     Spd=D(2)/Dh
200     R1(1,1)=Cpd
210     R1(1,2)=Spd
220     R1(2,1)=-Spd
230     R1(2,2)=Cpd
240     R1(3,3)=1
250     R2(1,1)=Ctd
260     R2(1,3)=-Std
270     R2(2,2)=1
280     R2(3,1)=Std
290     R2(3,3)=Ctd
300     MAT Rt= R2*R1
310   ELSE
320     MAT Rt= IDN
330   END IF
340   MAT Vtm= Rt*V
350   Vh=SQR(Vtm(1)*Vtm(1)+Vtm(2)*Vtm(2))
360   Vm=SQR(Vh*Vh+Vtm(3)*Vtm(3))
370   IF Vh/Vm>1.E-12 THEN
380     Cpv=Vtm(1)/Vh
390     Spv=Vtm(2)/Vh
400     R3(1,1)=Cpv
410     R3(1,2)=Spv
420     R3(3,1)=0
430     R3(2,1)=-Spv
440     R3(2,2)=Cpv
450     R3(3,3)=1
460   ELSE
470     MAT R3= IDN
480   END IF
490   MAT R= R3*Rt
500   Vt=Vh
510   Vr=Vtm(3)
520 SUBEND
```

```
10  SUB Geomfldpos(Xf(*),Rds(*),R,Ct,St,Cp,Sp)
20      !
30      !Transforms field point in general frame to special dipole-
40      !Sphere frame and converts it to polar coordinates in that
50      !Frame.
60      !
70    OPTION BASE 1
80    DIM X(3)
90    MAT X= Rds*Xf
100   Rh=SQR(X(1)*X(1)+X(2)*X(2))
```

```
110   R=SQR(Rh*Rh+X(3)*X(3))
120   St=Rh/R
130   Ct=X(3)/R
140   IF St>1.E-12 THEN
150     Cp=X(1)/Rh
160     Sp=X(2)/Rh
170   ELSE
180     Cp=1
190     Sp=0
200   END IF
210  SUBEND
```

```
10  SUB Geomfldb_e(Ct,St,Cp,Sp,Bp(*),Ep(*),Rds(*),B(*),E(*))
20     !
30     !Transforms complex polar coordinate fields in the special
40     !Dipole-sphere frame to cartesian coordinates in that frame
50     !And then rotates the fields back to the original general
60     !Frame.
70     !
80    OPTION BASE 1
90    DIM Irds(3,3),Pc(3,3),Et(3,2),Bt(3,2)
100   MAT Irds= INV(Rds)
110   Pc(1,1)=St*Cp
120   Pc(1,2)=Ct*Cp
130   Pc(1,3)=-Sp
140   Pc(2,1)=St*Sp
150   Pc(2,2)=Ct*Sp
160   Pc(2,3)=Cp
170   Pc(3,1)=Ct
180   Pc(3,2)=-St
190   Pc(3,3)=0
200   MAT Bt= Pc*Bp
210   MAT Et= Pc*Ep
220   MAT B= Irds*Bt
230   MAT E= Irds*Et
240  SUBEND
```

```
10  SUB Jcomb(L,Ur,Ui,R,Jr,Ji,Jcr,Jci)
20    CALL Spherejnz(L,Ur,Ui,Jr,Ji)
30    CALL Spherejnz(L+1,Ur,Ui,J1r,J1i)
40    Jcr=((L+1)*Jr-(Ur*J1r-Ui*J1i))/R
50    Jci=((L+1)*Ji-(Ur*J1i+Ui*J1r))/R
60  SUBEND
```

```
10  SUB Hcomb(L,Ur,Ui,R,Hr,Hi,Hcr,Hci)
20    CALL Spherehnz(L,Ur,Ui,Hr,Hi)
30    CALL Spherehnz(L+1,Ur,Ui,H1r,H1i)
40    Hcr=((L+1)*Hr-(Ur*H1r-Ui*H1i))/R
50    Hci=((L+1)*Hi-(Ur*H1i+Ui*H1r))/R
60  SUBEND
```

```
10 SUB Spherejnz(N,Zr,Zi,Sjr,Sji)
20   CALL Jnuevrywhr(N+.5,Zr,Zi,Jr,Ji)
30   Dm=Zr*Zr+Zi*Zi
40   Ur=Zr/Dm
50   Ui=-Zi/Dm
60   Nm=SQR(Ur*Ur+Ui*Ui)
70   Sur=SQR((Nm+Ur)/2)
80   Sui=SGN(Ui)*SQR(ABS(Nm-Ur)/2)
90   Sjr=SQR(PI/2)*(Sur*Jr-Sui*Ji)
100  Sji=SQR(PI/2)*(Sur*Ji+Sui*Jr)
110 SUBEND
```

```
10 SUB Spherehnz(N,Zr,Zi,Shr,Shi)
20   CALL Hnuevrywhr(N+.5,Zr,Zi,Hr,Hi)
30   Dm=Zr*Zr+Zi*Zi
40   Ur=Zr/Dm
50   Ui=-Zi/Dm
60   Nm=SQR(Ur*Ur+Ui*Ui)
70   Sur=SQR((Nm+Ur)/2)
80   Sui=SGN(Ui)*SQR(ABS(Nm-Ur)/2)
90   Shr=SQR(PI/2)*(Sur*Hr-Sui*Hi)
100  Shi=SQR(PI/2)*(Sur*Hi+Sui*Hr)
110 SUBEND
```

```
10 SUB Jnuevrywhr(Nu,Zr,Zi,Jr,Ji)
20   Zmag=SQR(Zr*Zr+Zi*Zi)
30   IF Nu<10 THEN
40     IF Zmag<10 THEN
50       CALL Jnu(Nu,Zr,Zi,Jr,Ji,1.E-28)
60     ELSE
70       IF Nu=0 THEN
80         CALL Jnasy(Nu,Zr,Zi,Jr,Ji,5)
90       ELSE
100        CALL Jfiuniasym(Nu,Zr,Zi,Jr,Ji,5)
110      END IF
120    END IF
130  ELSE
140    CALL Jfiuniasym(Nu,Zr,Zi,Jr,Ji,5)
150  END IF
160 SUBEND
```

```
10 SUB Hnuevrywhr(Nu,Zr,Zi,Hr,Hi)
20   Zmag=SQR(Zr*Zr+Zi*Zi)
30   IF Nu<10 THEN
40     IF Zmag<10 THEN
50       CALL H1nu(Nu,Zr,Zi,Hr,Hi,1.E-28)
60     ELSE
70       IF Nu=0 THEN
80         CALL H1nasy(Nu,Zr,Zi,Hr,Hi,5)
90       ELSE
100        CALL Hfkuniasym(Nu,Zr,Zi,Hr,Hi,5)
110      END IF
```

```
120     END IF
130    ELSE
140     CALL Hfkuniasym(Nu,Zr,Zi,Hr,Hi,5)
150    END IF
160  SUBEND

10   SUB Pl_pldot(N,X,Pl(*),Pldot(*))
20    Pl(1)=1
30    IF N=0 THEN 60
40    Pl(2)=X
50    IF N=1 THEN 80
60    Pldot(1)=0
70    IF N=0 THEN 170
80    Pldot(2)=1
90    IF N=1 THEN 170
100    FOR L=3 TO N+1
110     C1=1/(L-1)
120     C2=1-C1
130     C3=2-C1
140     Pl(L)=C3*X*Pl(L-1)-C2*Pl(L-2)
150     Pldot(L)=(C3*X*Pldot(L-1)-Pldot(L-2))/C2
160    NEXT L
170  SUBEND

10   SUB Nde(Lmax,K1r,K1i,K2r,K2i,Mur,R,Ndea(*))
20    U1r=K1r*R
30    U1i=K1i*R
40    U2r=K2r*R
50    U2i=K2i*R
60    FOR L=1 TO Lmax
70     ON ERROR GOTO 110
80     CALL Spherejnz(L-1,U1r,U1i,J1r,J1i)
90     CALL Spherejnz(L,U1r,U1i,J11r,J11i)
100     GOTO 140
110     J1ratr=0
120     J1rati=1
130     GOTO 190
140     Dm=J1r*J1r+J1i*J1i
150     Rpr=J1r/Dm
160     Rpi=-J1i/Dm
170     J1ratr=Rpr*J11r-Rpi*J11i
180     J1rati=Rpr*J11i+Rpi*J11r
190     OFF ERROR
200     CALL Spherejnz(L-1,U2r,U2i,J2r,J2i)
210     CALL Spherejnz(L,U2r,U2i,J21r,J21i)
220     CALL Spherehnz(L-1,U2r,U2i,H2r,H2i)
230     CALL Spherehnz(L,U2r,U2i,H21r,H21i)
240     Nb1r=L*J2r-(U2r*J21r-U2i*J21i)
250     Nb1i=L*J2i-(U2r*J21i+U2i*J21r)
260     Nb2r=L-(U1r*J1ratr-U1i*J1rati)
270     Nb2i=-(U1r*J1rati+U1i*J1ratr)
280     Topr=Mur*Nb1r-(J2r*Nb2r-J2i*Nb2i)
```

```
290      Topi=Mur*Nb1i-(J2r*Nb2i+J2i*Nb2r)
300      Db1r=L*H2r-(U2r*H21r-U2i*H21i)
310      Db1i=L*H2i-(U2r*H21i+U2i*H21r)
320      Btmr=Mur*Db1r-(H2r*Nb2r-H2i*Nb2i)
330      Btmi=Mur*Db1i-(H2r*Nb2i+H2i*Nb2r)
340      Bm=Btmr*Btmr+Btmi*Btmi
350      Rbmr=Btmr/Bm
360      Rbmi=-Btmi/Bm
370      Ndea(L,1)=-(Topr*Rbmr-Topi*Rbmi)
380      Ndea(L,2)=-(Topr*Rbmi+Topi*Rbmr)
390    NEXT L
400  SUBEND

10  SUB Ndm(Lmax,K1r,K1i,K2r,K2i,Mur,R,Ndma(*))
20    U1r=K1r*R
30    U1i=K1i*R
40    U2r=K2r*R
50    U2i=K2i*R
60    K1sqr=K1r*K1r-K1i*K1i
70    K1sqi=2*K1r*K1i
80    K2sqr=K2r*K2r-K2i*K2i
90    K2sqi=2*K2r*K2i
100   Dm=K1r*K1r+K1i*K1i
110   Rpr=K1r/Dm
120   Rpi=-K1i/Dm
130   Ep2r=Rpr*K2r-Rpi*K2i
140   Ep2i=Rpr*K2i+Rpi*K2r
150   FOR L=1 TO Lmax
160     ON ERROR GOTO 200
170     CALL Spherejnz(L-1,U1r,U1i,J1r,J1i)
180     CALL Spherejnz(L,U1r,U1i,J11r,J11i)
190     GOTO 230
200     J1ratr=0
210     J1rati=1
220     GOTO 280
230     Dm=J1r*J1r+J1i*J1i
240     Rpr=J1r/Dm
250     Rpi=-J1i/Dm
260     J1ratr=Rpr*J11r-Rpi*J11i
270     J1rati=Rpr*J11i+Rpi*J11r
280     OFF ERROR
290     CALL Spherejnz(L-1,U2r,U2i,J2r,J2i)
300     CALL Spherejnz(L,U2r,U2i,J21r,J21i)
310     CALL Spherehnz(L-1,U2r,U2i,H2r,H2i)
320     CALL Spherehnz(L,U2r,U2i,H21r,H21i)
330     Nb1r=L*J2r-(U2r*J21r-U2i*J21i)
340     Nb1i=L*J2i-(U2r*J21i+U2i*J21r)
350     Nb2r=L-(U1r*J1ratr-U1i*J1rati)
360     Nb2i=-(U1r*J1rati+U1i*J1ratr)
370     Nfacr=Mur*(Ep2r*J2r-Ep2i*J2i)
380     Nfaci=Mur*(Ep2r*J2i+Ep2i*J2r)
390     Topr=Nb1r-(Nfacr*Nb2r-Nfaci*Nb2i)
```

```
400     Topi=Nb1i-(Nfacr*Nb2i+Nfaci*Nb2r)
410     Db1r=L*H2r-(U2r*H21r-U2i*H21i)
420     Db1i=L*H2i-(U2r*H21i+U2i*H21r)
430     Dfacr=Mur*(Ep2r*H2r-Ep2i*H2i)
440     Dfaci=Mur*(Ep2r*H2i+Ep2i*H2r)
450     Btmr=Db1r-(Dfacr*Nb2r-Dfaci*Nb2i)
460     Btmi=Db1i-(Dfacr*Nb2i+Dfaci*Nb2r)
470     Bm=Btmr*Btmr+Btmi*Btmi
480     Rbmr=Btmr/Bm
490     Rbmi=-Btmi/Bm
500     Ndma(L,1)=-(Topr*Rbmr-Topi*Rbmi)
510     Ndma(L,2)=-(Topr*Rbmi+Topi*Rbmr)
520   NEXT L
530  SUBEND

10  SUB Ndeicon(Lmax,K2r,K2i,R,Ndea(*))
20    U2r=K2r*R
30    U2i=K2i*R
40    FOR L=1 TO Lmax
50      CALL Spherejnz(L-1,U2r,U2i,J2r,J2i)
60      CALL Spherehnz(L-1,U2r,U2i,H2r,H2i)
70      Den=H2r*H2r+H2i*H2i
80      Ih2r=H2r/Den
90      Ih2i=-H2i/Den
100     Ndea(L,1)=-(J2r*Ih2r-J2i*Ih2i)
110     Ndea(L,2)=-(J2r*Ih2i+J2i*Ih2r)
120   NEXT L
130 SUBEND

140 SUB Ndmicon(Lmax,K2r,K2i,R,Ndma(*))
150   U2r=K2r*R
160   U2i=K2i*R
170   FOR L=1 TO Lmax
180     CALL Spherejnz(L-1,U2r,U2i,J2r,J2i)
190     CALL Spherejnz(L,U2r,U2i,J21r,J21i)
200     CALL Spherehnz(L-1,U2r,U2i,H2r,H2i)
210     CALL Spherehnz(L,U2r,U2i,H21r,H21i)
220     A=U2r*H21r-U2i*H21i
230     B=U2r*H21i+U2i*H21r
240     C=H2r*L-A
250     D=H2i*L-B
260     Den=C*C+D*D
270     Ihr=C/Den
280     Ihi=-D/Den
290     A=U2r*J21r-U2i*J21i
300     B=U2r*J21i+U2i*J21r
310     C=J2r*L-A
320     D=J2i*L-B
330     Ndma(L,1)=-(C*Ihr-D*Ihi)
340     Ndma(L,2)=-(C*Ihi+D*Ihr)
350   NEXT L
360 SUBEND
```

SERIES REPRESENTATIONS($\nu < 10, |z| < 10$)

The Bessel functions of the first kind $J_\nu(z)$ can be constructed from the series representation

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(-\frac{z^2}{4}\right)^k}{k!\Gamma(\nu+k+1)}. \tag{A1}$$

The Hankel function, or Bessel function of the third kind for "outgoing waves", $H_\nu^{(1)}(z)$ is related to $J_\nu(z)$ and the Bessel functions of the second kind, $Y_\nu(z)$, by

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z). \tag{A2}$$

For non-integral values of ν, $Y_\nu(z)$ can be defined by

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)} \tag{A3}$$

but for integer values of ν, it is necessary to use the more complicated expression

$$Y_n(z) = -\frac{\left(\frac{z}{2}\right)^{-n}}{\pi} \sum_{k=0}^{n-1} (n-k-1)\frac{!}{k!}\left(\frac{z^2}{4}\right)^k + \frac{2}{\pi}\ln\left(\frac{z}{2}\right)J_n(z)$$

$$-\frac{\left(\frac{z}{2}\right)^{n}}{\pi} \sum_{k=0}^{\infty} \{\psi(k+1)+\psi(n+k+1)\}\frac{\left(-\frac{z^2}{4}\right)^k}{k!(n+k)!} \tag{A4}$$

where $\psi(n)$ is given by

$$\psi(1) = -\gamma \quad , \quad \psi(n) = -\gamma + \sum_{k=1}^{n-1}\frac{1}{k} \quad (n \geq 2), \tag{A5}$$

and γ is the Euler constant 0.5772156649015328606065l2.

In the following seven subroutines, the functions $J_\nu(z)$ and $H_\nu^{(1)}(z)$ are generated. Use of these routines has been restricted to cases where the magnitudes of ν and z do not exceed 10. For larger values of ν and z, various asymptotic techniques are used. The discussion of these begins below, after the series expansion routines.

```
10 SUB Jn(N,Zzr,Zzi,Jnr,Jni,Err)
20    !N is the (Integer) order.
30    !Zzr and Zzi are the real and imaginary parts of the argument.
40    !Jnr and Jni are the real and imaginary parts of the Bessel function.
50    !Err is the relative truncation error in the truncation of the series
60    !expansion of the Bessel function.
70   M=0
80   Zr=Zzr/2
90   Zi=Zzi/2
100  U2=Zi*Zi-Zr*Zr
110  V2=-2*Zr*Zi
120  IF N=0 THEN
130    Tr=1
140    Ti=0
150    Dr=Tr
160    Di=Ti
170  ELSE
180    U=1
190    V=0
200    FOR L=1 TO N
210      Ut=U*Zr-V*Zi
220      Vt=U*Zi+V*Zr
230      U=Ut/L
240      V=Vt/L
250    NEXT L
260    Tr=U
270    Ti=V
280    Dr=Tr
290    Di=Ti
300  END IF
310  M=M+1
320  Ttr=(Tr*U2-Ti*V2)/M/(M+N)
330  Tti=(Tr*V2+Ti*U2)/M/(M+N)
340  Tr=Ttr
350  Ti=Tti
360  Dr=Dr+Tr
370  Di=Di+Ti
380  IF (Tr*Tr+Ti*Ti)/(Dr*Dr+Di*Di)<Err*Err THEN 400
390  GOTO 310
400  Jnr=Dr
410  Jni=Di
420 SUBEND
```

```
10 SUB H1n(N,Zzr,Zzi,H1nr,H1ni,Err)
20    !N is the (Integer) order
30    !Zzr and Zzi are the real and imaginary parts of the argument.
40    !H1nr and H1ni are the real and imaginary parts of the Bessel
50    !function. Err is the relative truncation error in the truncation
60    !of the series expansion of the Bessel function.
70   M=0
```

```
80    Zr=Zzr/2
90    Zi=Zzi/2
100    P1=0
110    P2=0
120    CALL Prinlogz(Zr,Zi,Lr,Li)
130    Gam=.577215664901533
140    Cr=PI-2*Li
150    Cii=2*(Gam+Lr)
160    U2=Zi*Zi-Zr*Zr
170    V2=-2*Zr*Zi
180    IF N=0 THEN
190      Sr=0
200      Si=0
210      Pwr=1
220      Pwi=0
230      Dr=Cr
240      Di=Cii
250    ELSE
260      U=1
270      V=0
280      Nm=1
290      FOR L=1 TO N
300        Ut=U*Zr-V*Zi
310        Vt=U*Zi+V*Zr
320        U=Ut/L
330        V=Vt/L
340        P2=P2+1/L
350        Nm=Nm*L
360      NEXT L
370      Dn=U*U+V*V
380      Uu=-V/Dn/Nm
390      Vv=-U/Dn/Nm
400      Nm=Nm/N
410      Pwr=U
420      Pwi=V
430      Sr=Nm*Uu
440      Si=Nm*Vv
450      Dr=Cr*Pwr-(Cii-P2)*Pwi+Sr
460      Di=Cr*Pwi+(Cii-P2)*Pwr+Si
470    END IF
480    M=M+1
490    P1=P1+1/M
500    P2=P2+1/(N+M)
510    Pwrt=(Pwr*U2-Pwi*V2)/M/(M+N)
520    Pwit=(Pwr*V2+Pwi*U2)/M/(M+N)
530    Pwr=Pwrt
540    Pwi=Pwit
550    Ci=Cii-P1-P2
560    Tr=Pwr*Cr-Pwi*Ci
570    Ti=Pwi*Cr+Pwr*Ci
580    Dr=Dr+Tr
590    Di=Di+Ti
```

```
600  IF M<N THEN
610    Ssr=-(Sr*U2-Si*V2)/M/(N-M)
620    Ssi=-(Sr*V2+Si*U2)/M/(N-M)
630    Sr=Ssr
640    Si=Ssi
650  ELSE
660    Sr=0
670    Si=0
680  END IF
690  Dr=Dr+Sr
700  Di=Di+Si
710  IF (Tr*Tr+Ti*Ti)/(Dr*Dr+Di*Di)<Err*Err THEN 730
720  GOTO 480
730  H1nr=Dr/PI
740  H1ni=Di/PI
750 SUBEND
```

```
10  SUB Jnu(Nu,Zzr,Zzi,Jnur,Jnui,Err)
20    !Nu is the order
30    !Zzr and Zzi are the real and imaginary parts of the argument.
40    !Jnur and Jnui are the real and imaginary parts of the Bessel
50    !function. Err is the relative truncation error in the truncation
60    !of the series expansion of the Bessel function.
70   M=0
80   Zr=Zzr/2
90   Zi=Zzi/2
100  U2=Zi*Zi-Zr*Zr
110  V2=-2*Zr*Zi
120  U=1
130  V=0
140  Dn=1
150  Dr=1
160  Di=0
170  M=M+1
180  Ut=U*U2-V*V2
190  Vt=U*V2+V*U2
200  U=Ut
210  V=Vt
220  Dn=Dn*(Nu+M)*M
230  Tr=U/Dn
240  Ti=V/Dn
250  Dr=Dr+Tr
260  Di=Di+Ti
270  IF (ABS(Tr)+ABS(Ti))/(ABS(Dr)+ABS(Di))<Err THEN 290
280  GOTO 170
290  Jnur=Dr
300  Jnui=Di
310  N=INT(Nu)
320  Eps=Nu-N
330  CALL Gamma(Nu+1,G0)
```

```
340   CALL Prinlogz(Zr,Zi,Lzr,Lzi)
350   Mo=EXP(Eps*Lzr)
360   Ph=Eps*Lzi
370   Cr=Mo*COS(Ph)
380   Ci=Mo*SIN(Ph)
390   M=0
400   Jr=(Cr*Jnur-Ci*Jnui)/G0
410   Ji=(Cr*Jnui+Ci*Jnur)/G0
420   U=1
430   V=0
440   IF N<0 THEN 570
450   IF N=0 THEN 670
460   U=Zr
470   V=Zi
480   M=M+1
490   IF M=N THEN 670
500   Ut=U*Zr-V*Zi
510   Vt=U*Zi+V*Zr
520   U=Ut
530   V=Vt
540   M=M+1
550   IF M=N THEN 670
560   GOTO 500
570   Ut=U*Zr-V*Zi
580   Vt=U*Zi+V*Zr
590   U=Ut
600   V=Vt
610   M=M-1
620   IF M=N THEN 640
630   GOTO 570
640   Dnm=U*U+V*V
650   U=U/Dnm
660   V=-V/Dnm
670   Jnur=U*Jr-V*Ji
680   Jnui=U*Ji+V*Jr
690  SUBEND
```

```
10   SUB H1nu(Nu,Zr,Zi,H1nur,H1nui,Err)
20     !See Jnu.
30    CALL Jnu(Nu,Zr,Zi,Jr,Ji,Err)
40    CALL Jnu(-Nu,Zr,Zi,Jmr,Jmi,Err)
50    C=COS(Nu*PI)
60    S=SIN(Nu*PI)
70    H1nur=Jr-(C*Ji-Jmi)/S
80    H1nui=Ji+(C*Jr-Jmr)/S
90   SUBEND
```

```
10  SUB Prinlogz(Zr,Zi,Lzr,Lzi)
20    Lzr=LOG(Zr*Zr+Zi*Zi)/2
30    CALL Atn2(Zr,Zi,Lzi)
40  SUBEND
```

```
10  SUB Atn2(X,Y,Theta)
20    IF X=0 THEN
30      IF Y<0 THEN
40        Theta=-PI/2
50      ELSE
60        Theta=PI/2
70      END IF
80      GOTO 200
90    END IF
100   Theta=ATN(Y/X)
110   IF X>=0 THEN
120     Theta=Theta
130   ELSE
140     IF Y>=0 THEN
150       Theta=Theta+PI
160     ELSE
170       Theta=Theta-PI
180     END IF
190   END IF
200 SUBEND
```

```
10  SUB Gamma(X,G)
20    OPTION BASE 1
30  !
40  !Computes Gamma(x) for any real x(where defined).
50  !Precision is 1E-12
60  !
70    DIM A(20),Z(20)
80    IF X=1 THEN
90      G=1
100     GOTO 540
110   END IF
120   IF X>=1 THEN
130     Xx=X
140     Y=1
150     Xx=Xx-1
160     IF Xx>=0 AND Xx<=1 THEN GOTO 280
170     Y=Xx*Y
180     Xx=Xx-1
190     GOTO 160
200   ELSE
210     Xx=X-1
220     Y=1
```

```
230      Xx=Xx+1
240      Y=Y/Xx
250      IF Xx>=0 AND Xx<=1 THEN 280
260      GOTO 230
270    END IF
280    A(1)=1
290    A(2)=.57721566490
300    A(3)=-.65587807152
310    A(4)=-.04200263503
320    A(5)=.16653861138
330    A(6)=-.04219773456
340    A(7)=-.00962197153
350    A(8)=.00721894325
360    A(9)=-.00116516759
370    A(10)=-.00021524167
380    A(11)=.00012805028
390    A(12)=-.00002013485
400    A(13)=-.00000125049
410    A(14)=.00000113303
420    A(15)=-.00000020563
430    A(16)=.00000000612
440    A(17)=.00000000500
450    A(18)=-.00000000118
460    A(19)=.00000000010
470    A(20)=.00000000001
480    Z(1)=Xx
490    FOR I=2 TO 20
500      Z(I)=Xx*Z(I-1)
510    NEXT I
520    Gi=DOT(A,Z)
530    G=Y*Xx/Gi
540  SUBEND
```

ASYMPTOTIC REPRESENTATIONS ($\nu < 10, |z| > 10$)

For $\nu < 10$ and large values of z, the following asymptotic expansions are useful

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\{P(\nu,z)\cos\chi - Q(\nu,z)\sin\chi\} \tag{A6}$$

and

$$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}}\{P(\nu,z) + iQ(\nu,z)\}e^{i\chi} \tag{A7}$$

where $\chi = z - \left(\frac{1}{2}\nu + \frac{1}{2}\right)\pi$ and, with $\mu = 4\nu^2$, the functions $P(\nu,z)$ and $Q(\nu,z)$ are given by

$$P(\nu,z) \sim \sum_{k=0}^{\infty} (-1)^k \frac{(\nu,2k)}{(2z)^{2k}} \tag{A8}$$

$$Q(\nu,z) \sim \sum_{k=0}^{\infty} (-1)^k \frac{(\nu,2k+1)}{(2z)^{2k+1}} \tag{A9}$$

and

$$(\nu,0)=1 \quad , \quad (\nu,2)=(\mu-1)\frac{(\mu-9)}{2!} \quad , \quad (\nu,4)=(\mu-1)(\mu-9)(\mu-25)\frac{(\mu-49)}{4!} \quad \cdots \tag{A10}$$

and

$$(\nu,1)=\mu-1 \quad , \quad (\nu,3)=(\mu-1)(\mu-9)\frac{(\mu-25)}{3!} \quad \cdots \quad . \tag{A11}$$

The next four subroutines apply these expressions to the calculation of the Bessel functions.

```
10  SUB Jnasy(N,Zr,Zi,Jnr,Jni,Trms)
20    !Note: Valid for arbitrary real N
30    !Zr and Zi real and imaginary parts of argument
40    !Jnr and Jni real and imaginary parts of the Bessel function.
50    !Trms is the number of terms retained in the asymptotic expansion.
60    CALL Pnz(N,Zr,Zi,Pnr,Pni,Trms)
70    CALL Qnz(N,Zr,Zi,Qnr,Qni,Trms)
80    Chir=Zr-(2*N+1)*PI/4
90    Chii=Zi
100   E=EXP(Chii)/2
110   Ei=1/4/E
120   C=COS(Chir)
130   S=SIN(Chir)
140   Cor=C*(E+Ei)
150   Coi=-S*(E-Ei)
160   Sir=S*(E+Ei)
170   Sii=C*(E-Ei)
180   Utlr=Pnr*Cor-Pni*Coi-Qnr*Sir+Qni*Sii
190   Utli=Pnr*Coi+Pni*Cor-Qnr*Sii-Qni*Sir
200   Zm=SQR(Zr*Zr+Zi*Zi)
210   Aa=SQR((Zm+Zr)/2)
220   Bb=SQR((Zm-Zr)/2)
230   Cc=Aa*Aa+Bb*Bb
240   Dd=SQR(2/PI)/Cc
250   Jnr=Dd*(Utlr*Aa+Utli*Bb)
260   Jni=Dd*(-Utlr*Bb+Utli*Aa)
270 SUBEND
```

```
10   SUB H1nasy(N,Zr,Zi,H1nr,H1ni,Trms)
20     !Note: Valid for arbitrary real N
30     !Zr and Zi real and imaginary parts of argument
40     !H1nr and H1ni real and imaginary parts of the Bessel function.
50     !Trms is the number of terms retained in the asymptotic expansion.
60    CALL Pnz(N,Zr,Zi,Pnr,Pni,Trms)
70    CALL Qnz(N,Zr,Zi,Qnr,Qni,Trms)
80    Chir=Zr-(2*N+1)*PI/4
90    Chii=Zi
100   E=EXP(-Chii)
110   C=COS(Chir)
120   S=SIN(Chir)
130   Utlr=(Pnr-Qni)*C-(Pni+Qnr)*S
140   Utli=(Pnr-Qni)*S+(Pni+Qnr)*C
150   Zm=SQR(Zr*Zr+Zi*Zi)
160   Aa=SQR((Zm+Zr)/2)
170   Bb=SGN(Zi)*SQR((Zm-Zr)/2)
180   Cc=Aa*Aa+Bb*Bb
190   Dd=E*SQR(2/PI)/Cc
200   H1nr=Dd*(Utlr*Aa+Utli*Bb)
210   H1ni=Dd*(-Utlr*Bb+Utli*Aa)
220  SUBEND
```

```
10   SUB Pnz(N,Zr,Zi,Pnr,Pni,Trms)
20     !Used with Jnasy and H1nasy.
30    M=0
40    Mu=4*N*N
50    Tr=1
60    Ti=0
70    Dr=Tr
80    Di=Ti
90    U2=64*(Zr*Zr-Zi*Zi)
100   V2=128*Zr*Zi
110   Dn=U2*U2+V2*V2
120   U=-U2/Dn
130   V=V2/Dn
140   M=M+2
150   Ttr=Tr*U-Ti*V
160   Tti=Tr*V+Ti*U
170   Tr=Ttr/M/(M-1)
180   Ti=Tti/M/(M-1)
190   Fac=1
200   FOR I=1 TO 2*M-1 STEP 2
210     Fac=Fac*(Mu-I*I)
220   NEXT I
230   Dr=Dr+Tr*Fac
240   Di=Di+Ti*Fac
250   IF M/2+1<Trms THEN 140
```

```
260   Pnr=Dr
270   Pni=Di
280  SUBEND
```

```
10   SUB Qnz(N,Zr,Zi,Qnr,Qni,Trms)
20     !Used with Jnasy and H1nasy.
30    M=1
40    Mu=4*N*N
50    Dnm=8*(Zr*Zr+Zi*Zi)
60    Tr=Zr/Dnm
70    Ti=-Zi/Dnm
80    Dr=Tr*(Mu-1)
90    Di=Ti*(Mu-1)
100   U2=64*(Zr*Zr-Zi*Zi)
110   V2=128*Zr*Zi
120   Dn=U2*U2+V2*V2
130   U=-U2/Dn
140   V=V2/Dn
150   M=M+2
160   Ttr=Tr*U-Ti*V
170   Tti=Tr*V+Ti*U
180   Tr=Ttr/M/(M-1)
190   Ti=Tti/M/(M-1)
200   Fac=1
210   FOR I=1 TO 2*M-1 STEP 2
220     Fac=Fac*(Mu-I*I)
230   NEXT I
240   Dr=Dr+Tr*Fac
250   Di=Di+Ti*Fac
260   IF (M+1)/2<Trms THEN 150
270   Qnr=Dr
280   Qni=Di
290  SUBEND
```

UNIFORM SYMPTOTIC REPRESENTATION $(\nu > 10, |z| > 10, \nu > |z|)$

Abramowitz and Stegun[A1] give uniform asymptotic expansions for the modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$. These functions can be used to generate the functions $J_\nu(z)$ and $H_\nu^{(1)}(z)$ by means of the identities

$$J_\nu(z) = e^{\frac{i\nu\pi}{2}} I_\nu\left(z e^{\frac{-i\pi}{2}}\right) \tag{A12}$$

and

$$H_\nu^{(1)}(z) = \frac{-2i}{\pi} e^{\frac{-i\nu\pi}{2}} K_\nu\left(z e^{\frac{-i\pi}{2}}\right) \tag{A13}$$

where the argument of z is restricted to $-\pi/2 < \arg z < \pi$. This restriction causes no problem, since the argument of z in electromagnetic applications is limited to $0 \leq \arg z < \pi/2$.

The uniform asymptotic expansions for $I_\nu(z)$ and $K_\nu(z)$ are given by ($w = z/\nu$)

$$I_\nu(z) \sim \frac{e^{\nu\eta}}{\sqrt{2\pi\nu\sqrt{1+w^2}}}\left\{1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k}\right\} \tag{A14}$$

and

$$K_\nu(z) \sim \frac{\pi e^{-\nu\eta}}{\sqrt{2\pi\nu\sqrt{1+w^2}}}\left\{1 + \sum_{k=1}^{\infty} \frac{(-1)^k u_k(t)}{\nu^k}\right\} \tag{A15}$$

where

$$t = \frac{1}{\sqrt{1+w^2}} \quad \text{and} \quad \eta = \frac{1}{t} + \ln\left(\frac{w}{1+\frac{1}{t}}\right). \tag{A16}$$

The asymptotic representation of the product of J_ν and $H_\nu^{(1)}$ can be constructed from the product of their asymptotic representations, and this will help avoid overflow and underflow problems in computations. The product is

$$J_\nu(z)H_{\nu'}^{(1)}(z') = \frac{-2i}{\pi} e^{\frac{i(\nu-\nu')\pi}{2}} I_\nu\left(ze^{\frac{-i\pi}{2}}\right) K_{\nu'}\left(z'e^{\frac{-i\pi}{2}}\right) \tag{A17}$$

where

$$I_\nu(z)K_{\nu'}(z') \sim \frac{e^{(\nu\eta-\nu'\eta')}}{2\sqrt{\nu\nu'\sqrt{(1+w^2)(1+w'^2)}}}\left\{1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k}\right\}\left\{1 + \sum_{k=1}^{\infty} \frac{(-1)^k u_k(t')}{\nu'^k}\right\}. \tag{A18}$$

The functions $u_k(t)$ are defined by

$$u_0(t) = 1 \tag{A19}$$

and

$$u_{k+1}(t) = \frac{1}{2}t^2(1-t^2)u_k(t) + \frac{1}{8}\int_0^t d\tau(1-\tau^2)u_k(\tau). \tag{A20}$$

Abramowitz and Stegun[A1] give the functions u_k for values of $k = 0$ thru 4 and give references for $k = 5$ and 6. The following is a general method for generating u_k for any k. Use the representation

$$u_k(t) = a_{k,0}t^k + a_{k,1}t^{k+2} + \ldots + a_{k,i}t^{k+2i} + \ldots a_{k,k}t^{k+2k}. \tag{A21}$$

Then the coefficients are given by the following recursion formulas

$$a_{k+1,0} = a_{k,0}\left[\frac{k}{2} + \frac{1}{8(k+1)}\right] \tag{A22}$$

$$a_{k+1,i} = a_{k,i}\left[\frac{k+2i}{2} + \frac{1}{8(k+1+2i)}\right] - a_{k,i-1}\left[\frac{k+2(i-1)}{2} + \frac{5}{8(k+1+2i)}\right] \tag{A23}$$

and

$$a_{k+1,k+1} = -a_{k,k}\left[\frac{k+2k}{2} + \frac{5}{8(k+1+2\{k+1\})}\right] \tag{A24}$$

where $i = 1, 2, \ldots, k$.

```
10  SUB Jfiuniasym(Nu,Zr,Zi,Jr,Ji,Trms)
20    !Nu is the order.
30    !Zr and Zi real and imaginary parts of argument
40    !Jr and Ji real and imaginary parts of the Bessel function.
50    !Trms is the number of terms retained in the asymptotic expansion.
60   DIM A(10,10),U(10,2),T(30,2)
70   REDIM A(Trms,Trms),U(Trms,2),T(3*Trms,2)
80   Wr=Zi/Nu
90   Wi=-Zr/Nu
100  Ur=1+Wr*Wr-Wi*Wi
110  Ui=2*Wr*Wi
120  Um=SQR(Ur*Ur+Ui*Ui)
130  Radr=SQR((Um+Ur)/2)
140  Radi=SGN(Ui)*SQR((Um-Ur)/2)
150  Radm=Radi*Radi+Radr*Radr
160  Sradm=SQR(Radm)
170  Radrtr=SQR((Sradm+Radr)/2)
180  Radrti=SGN(Radi)*SQR((Sradm-Radr)/2)
190  Radrtm=Radrtr*Radrtr+Radrti*Radrti
200  Iradrtr=Radrtr/Radrtm
210  Iradrti=-Radrti/Radrtm
220  Tr=Radr/Radm
230  Ti=-Radi/Radm
240  Dnr=1+Radr
250  Dni=Radi
260  Dnm=Dnr*Dnr+Dni*Dni
270  Nmr=Dnr/Dnm
280  Nmi=-Dni/Dnm
```

```
250   Dni=Radi
260   Dnm=Dnr*Dnr+Dni*Dni
270   Nmr=Dnr/Dnm
280   Nmi=-Dni/Dnm
290   Argr=Wr*Nmr-Wi*Nmi
300   Argi=Wr*Nmi+Wi*Nmr
310   CALL Prinlogz(Argr,Argi,Lgr,Lgi)
320   Etar=Radr+Lgr
330   Etai=Radi+Lgi
340   CALL Ukcoefs(Trms,A(*))
350   CALL Uk(Tr,Ti,Trms,A(*),T(*),U(*))
360   Dumr=0
370   Dumi=0
380   Nufac=Nu
390   FOR I=1 TO Trms
400     Nufac=Nufac/Nu
410     Dumr=Dumr+U(I,1)*Nufac
420     Dumi=Dumi+U(I,2)*Nufac
430   NEXT I
440   E=EXP(Nu*Etar)
450   Co=COS(Nu*Etai)
460   Si=SIN(Nu*Etai)
470   Fac=1/SQR(2*PI*Nu)
480   Facr=Fac*E*(Co*Iradrtr-Si*Iradrti)
490   Faci=Fac*E*(Co*Iradrti+Si*Iradrtr)
500   Inur=Facr*Dumr-Faci*Dumi
510   Inui=Facr*Dumi+Faci*Dumr
520   Cn=COS(Nu*PI/2)
530   Sn=SIN(Nu*PI/2)
540   Jr=Cn*Inur-Sn*Inui
550   Ji=Cn*Inui+Sn*Inur
560  SUBEND
```

```
10   SUB Hfkuniasym(Nu,Zr,Zi,H1r,H1i,Trms)
20     !Nu is the order.
30     !Zr and Zi real and imaginary parts of argument
40     !H1r and H1i real and imaginary parts of the Bessel function.
50     !Trms is the number of terms retained in the asymptotic expansion.
60    DIM A(10,10),U(10,2),T(30,2)
70    REDIM A(Trms,Trms),U(Trms,2),T(3*Trms,2)
80    Wr=Zi/Nu
90    Wi=-Zr/Nu
100   Ur=1+Wr*Wr-Wi*Wi
110   Ui=2*Wr*Wi
120   Um=SQR(Ur*Ur+Ui*Ui)
130   Radr=SQR((Um+Ur)/2)
140   Radi=SGN(Ui)*SQR((Um-Ur)/2)
150   Radm=Radi*Radi+Radr*Radr
160   Sradm=SQR(Radm)
170   Radrtr=SQR((Sradm+Radr)/2)
```

```
180   Radrti=SGN(Radi)*SQR((Sradm-Radr)/2)
190   Radrtm=Radrtr*Radrtr+Radrti*Radrti
200   Iradrtr=Radrtr/Radrtm
210   Iradrti=-Radrti/Radrtm
220   Tr=Radr/Radm
230   Ti=-Radi/Radm
240   Dnr=1+Radr
250   Dni=Radi
260   Dnm=Dnr*Dnr+Dni*Dni
270   Nmr=Dnr/Dnm
280   Nmi=-Dni/Dnm
290   Argr=Wr*Nmr-Wi*Nmi
300   Argi=Wr*Nmi+Wi*Nmr
310   CALL Prinlogz(Argr,Argi,Lgr,Lgi)
320   Etar=Radr+Lgr
330   Etai=Radi+Lgi
340   CALL Ukcoefs(Trms,A(*))
350   CALL Uk(Tr,Ti,Trms,A(*),T(*),U(*))
360   Dumr=0
370   Dumi=0
380   Nufac=-Nu
390   FOR I=1 TO Trms
400     Nufac=-Nufac/Nu
410     Dumr=Dumr+U(I,1)*Nufac
420     Dumi=Dumi+U(I,2)*Nufac
430   NEXT I
440   E=EXP(-Nu*Etar)
450   Co=COS(Nu*Etai)
460   Si=-SIN(Nu*Etai)
470   Fac=SQR(PI/2/Nu)
480   Facr=Fac*E*(Co*Iradrtr-Si*Iradrti)
490   Faci=Fac*E*(Co*Iradrti+Si*Iradrtr)
500   Knur=Facr*Dumr-Faci*Dumi
510   Knui=Facr*Dumi+Faci*Dumr
520   Cn=COS(Nu*PI/2)
530   Sn=-SIN(Nu*PI/2)
540   Jr=Cn*Knur-Sn*Knui
550   Ji=Cn*Knui+Sn*Knur
560   H1r=2*Ji/PI
570   H1i=-2*Jr/PI
580 SUBEND
```

```
10  SUB Jh1produni(Nu,Nup,Zr,Zi,Zpr,Zpi,Prodr,Prodi,Trms)
20      !Nu and Nup are the orders of the two Bessel functions in the
product.
30      !Zr and Zi are the real and imaginary parts of the argument of one
40      !factor and Zpr and Zpi are those for the other factor.
50      !Prodr and Prodi are the real and imaginary parts of the Bessel
60      !function product.
70      !Trms is the number of terms retained in the asymptotic expansion.
```

```
80    DIM A(10,10),U(10,2),T(30,2)
90    REDIM A(Trms,Trms),U(Trms,2),T(3*Trms,2)
100   Wr=Zi/Nu
110   Wi=-Zr/Nu
120   Ur=1+Wr*Wr-Wi*Wi
130   Ui=2*Wr*Wi
140   Um=SQR(Ur*Ur+Ui*Ui)
150   Radr=SQR((Um+Ur)/2)
160   Radi=SGN(Ui)*SQR((Um-Ur)/2)
170   Radm=Radi*Radi+Radr*Radr
180   Sradm=SQR(Radm)
190   Radrtr=SQR((Sradm+Radr)/2)
200   Radrti=SGN(Radi)*SQR((Sradm-Radr)/2)
210   Radrtm=Radrtr*Radrtr+Radrti*Radrti
220   Iradrtr=Radrtr/Radrtm
230   Iradrti=-Radrti/Radrtm
240   Tr=Radr/Radm
250   Ti=-Radi/Radm
260   Dnr=1+Radr
270   Dni=Radi
280   Dnm=Dnr*Dnr+Dni*Dni
290   Nmr=Dnr/Dnm
300   Nmi=-Dni/Dnm
310   Argr=Wr*Nmr-Wi*Nmi
320   Argi=Wr*Nmi+Wi*Nmr
330   CALL Prinlogz(Argr,Argi,Lgr,Lgi)
340   Etar=Radr+Lgr
350   Etai=Radi+Lgi
360   CALL Ukcoefs(Trms,A(*))
370   CALL Uk(Tr,Ti,Trms,A(*),T(*),U(*))
380   Dumr=0
390   Dumi=0
400   Nufac=Nu
410   FOR I=1 TO Trms
420     Nufac=Nufac/Nu
430     Dumr=Dumr+U(I,1)*Nufac
440     Dumi=Dumi+U(I,2)*Nufac
450   NEXT I
460   MAT U= (0)
470   Wr=Zpi/Nup
480   Wi=-Zpr/Nup
490   Ur=1+Wr*Wr-Wi*Wi
500   Ui=2*Wr*Wi
510   Um=SQR(Ur*Ur+Ui*Ui)
520   Radr=SQR((Um+Ur)/2)
530   Radi=SGN(Ui)*SQR((Um-Ur)/2)
540   Radm=Radi*Radi+Radr*Radr
550   Sradm=SQR(Radm)
560   Radrtr=SQR((Sradm+Radr)/2)
570   Radrti=SGN(Radi)*SQR((Sradm-Radr)/2)
580   Radrtm=Radrtr*Radrtr+Radrti*Radrti
590   Iradrtpr=Radrtr/Radrtm
```

```
600   Iradrtpi=-Radrti/Radrtm
610   Tr=Radr/Radm
620   Ti=-Radi/Radm
630   Dnr=1+Radr
640   Dni=Radi
650   Dnm=Dnr*Dnr+Dni*Dni
660   Nmr=Dnr/Dnm
670   Nmi=-Dni/Dnm
680   Argr=Wr*Nmr-Wi*Nmi
690   Argi=Wr*Nmi+Wi*Nmr
700   CALL Prinlogz(Argr,Argi,Lgr,Lgi)
710   Etapr=Radr+Lgr
720   Etapi=Radi+Lgi
730   CALL Uk(Tr,Ti,Trms,A(*),T(*),U(*))
740   Dumpr=0
750   Dumpi=0
760   Nufac=-Nup
770   FOR I=1 TO Trms
780     Nufac=-Nufac/Nup
790     Dumpr=Dumpr+U(I,1)*Nufac
800     Dumpi=Dumpi+U(I,2)*Nufac
810   NEXT I
820   Sumr=Dumr*Dumpr-Dumi*Dumpi
830   Sumi=Dumr*Dumpi+Dumi*Dumpr
840   Radsr=Iradrtr*Iradrtpr-Iradrti*Iradrtpi
850   Radsi=Iradrtr*Iradrtpi+Iradrti*Iradrtpr
860   Dumprodr=(Sumr*Radsr-Sumi*Radsi)/(2*SQR(Nu*Nup))
870   Dumprodi=(Sumr*Radsi+Sumi*Radsr)/(2*SQR(Nu*Nup))
880   Etargr=Nu*Etar-Nup*Etapr
890   Etargi=Nu*Etai-Nup*Etapi
900   E=EXP(Etargr)
910   Co=COS(Etargi)
920   Si=SIN(Etargi)
930   Ikprodr=E*(Dumprodr*Co-Dumprodi*Si)
940   Ikprodi=E*(Dumprodr*Si+Dumprodi*Co)
950   Cfr=2*SIN((Nu-Nup)*PI/2)/PI
960   Cfi=-2*COS((Nu-Nup)*PI/2)/PI
970   Prodr=Ikprodr*Cfr-Ikprodi*Cfi
980   Prodi=Ikprodr*Cfi+Ikprodi*Cfr
990  SUBEND
```

```
10   SUB Uk(Tr,Ti,K,A(*),T(*),Uk(*))
20     !Functions used in the uniform asymptotic expansion of the Bessel
30     !functions for large orders.  A(*) is a matrix generated by the
40     !program Ukcoefs.  T(*) and Uk(*) are output to the uniform asymp-
50     !totic Bessel function routines.
60    T(1,1)=Tr
70    T(1,2)=Ti
80    FOR I=2 TO 3*K
90      T(I,1)=T(I-1,1)*Tr-T(I-1,2)*Ti
```

```
100     T(I,2)=T(I-1,1)*Ti+T(I-1,2)*Tr
110    NEXT I
120    Uk(1,1)=1
130    Uk(1,2)=0
140    FOR I=2 TO K
150      FOR J=1 TO I
160        Uk(I,1)=Uk(I,1)+A(I,J)*T(2*J+I-3,1)
170        Uk(I,2)=Uk(I,2)+A(I,J)*T(2*J+I-3,2)
180      NEXT J
190    NEXT I
200  SUBEND
```

```
10   SUB Ukcoefs(K,A(*))
20     !K is the maximum index value.  A(*) is output to Uk.
30    A(1,1)=1
40    FOR L=0 TO K-2
50      A(L+2,1)=A(L+1,1)*(L/2+1/8/(L+1))
60      A(L+2,L+2)=-A(L+1,L+1)*(3*L/2+5/24/(L+1))
70      IF L>=1 THEN
80        FOR M=1 TO L
90          A(L+2,M+1)=A(L+1,M+1)*((L+2*M)/2+1/(8*(L+1+2*M)))
100          A(L+2,M+1)=A(L+2,M+1)-A(L+1,M)*((L+2*(M-1))/2+5/(8*(L+1+2*M)))
110        NEXT M
120      END IF
130    NEXT L
140  SUBEND
```

REFERENCES

A1. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, (Applied Mathematics Series 55, 1965), National Bureau of Standards.

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Commander, Naval Air Development Center, Warminster, PA 18974-5000	
Attn: Dr. Lloyd Bob	13
Commander, Naval Surface Weapons Center Detachment, White Oak Laboratory, 10901 New Hampshire Avenue, Silver Spring, MD 20903-5000	
Attn: Dr. John Holmes	14
Attn: Mr. John Stahl	15
Commanding Officer, Naval Surface Weapons Center, Dahlgren Laboratory, Dahlgren, VA 22448	16
Commander, Naval Underwater Systems Center, Newport, RI 02841-5047	17
Officer in Charge, Naval Underwater Systems Center, New London Laboratory, New London, CT 06320	18
Commander, Naval Ocean Systems Center, San Diego, CA 92152-5000	19
Commander, Naval Weapons Center, China Lake, CA 93555-6001	20
Commanding Officer, Naval Research Laboratory, Washington, DC 20375	21
Director, Defense Advanced Research Projects Agency, 1400 Wilson Blvd, Arlington, VA 22209	22
Administrator, Defense Technical Information Center, Cameron Station, Alexandria, VA 22304-6130	23-24